Minimax quantum-state discrimination

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(Received 7 April 2005; published 8 September 2005)

We derive the optimal measurement for quantum-state discrimination without *a priori* probabilities, i.e., in a minimax strategy instead of the usually considered Bayesian one. We consider both minimal-error and unambiguous discrimination problems, and provide the relation between the optimal measurements according to the two schemes. We show that there are instances in which the minimum risk cannot be achieved by an orthogonal measurement, and this is a common feature of the minimax estimation strategy.

I. INTRODUCTION

Since the pioneering work of Helstrom [1] on quantum hypothesis testing, the problem of discriminating nonorthogonal quantum states has received much attention [2], with some experimental verifications as well [3]. The most popular scenarios are the minimal-error probability discrimination [1], where each measurement outcome selects one of the possible states and the error probability is minimized, and the optimal unambiguous discrimination of linearly independent states [4], where unambiguity is paid by the possibility of getting inconclusive results from the measurement. The problem of discrimination has been addressed also for bipartite quantum states, with both global joint measurements and local measurements with classical communication [5]. The concept of distinguishability can be applied also to all physically allowed transformations of quantum states, and in fact, more recently, the problem of discrimination has been considered for unitary transformations [6] and more general quantum operations [7]. In all the above-mentioned discrimination problems, a Bayesian approach has always been considered, with given *a priori* probability distribution for the states (or operations) to be discriminated.

In this paper, we consider the problem of optimal discrimination of quantum states in the minimax approach. In this strategy no prior probabilities are given. The relevance of this approach is both conceptual, since for a frequentist statistician the *a priori* probabilities have no meaning, and practical, because the prior probabilities may be actually unknown, as in a noncooperative cryptographic scenario. We will derive the optimal measurement for minimax state discrimination for both minimal-error and unambiguous discrimination problems. We will also provide the relation between the optimal measurements according to the minimax and the Bayesian strategies. We will show that, quite unexpectedly, there are instances in which the minimum risk can

be achieved only by nonorthogonal positive operator-valued measurement (POVM), and this is a common feature of the minimax estimation strategy.

The paper is organized as follows. In Sec. II we pose the problem of discrimination of two quantum states in the minimax scenario. Such an approach is equivalent to a minimax problem, where one should maximize the smallest of the two probabilities of correct detection over all measurement schemes. For simplicity we will consider equal weights (i.e., equal prices of misidentifying the states), and we will provide the optimal measurement for the minimax discrimination, along with the connection with the optimal Bayesian solution. As mentioned, a striking result of this section is the existence of couples of mixed states for which the optimal minimax measurement is unique and *nonorthogonal*. In Sec. III we generalize the results for two-state discrimination to the case of $N \geqslant 2$ states and arbitrary weights. First, we consider the simplest situation of the covariant state discrimination problem. Then, we address the problem in generality, resorting to the related convex programming method. In Sec. IV we provide the solution of the minimax discrimination problem in the scenario of unambiguous discrimination. The conclusions of the paper are summarized in Sec. V.

II. OPTIMAL MINIMAX DISCRIMINATION OF TWO QUANTUM STATES

We are given two states ρ_1 and ρ_2, generally mixed, and we want to find the optimal measurement to discriminate between them in a minimax strategy. The measurement is described by a POVM with two outcomes, namely, $\vec{P} \equiv (P_1, P_2)$, where P_i for $i = 1, 2$ are nonnegative operators satisfying $P_1 + P_2 = I$.

In the usually considered Bayesian approach to the discrimination problem, the states are given with *a priori* probability distribution $\vec{a} \equiv (a_1, a_2)$, respectively, and one looks for the POVM that minimizes the average error probability

$$p_E = a_1 \mathrm{Tr}(\rho_1 P_2) + a_2 \mathrm{Tr}(\rho_2 P_1). \tag{1}$$

The solution can then be achieved by taking the orthogonal POVM made by the projectors on the support of the positive

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and negative parts of the Hermitian operator $a_1\rho_1 - a_2\rho_2$, and hence one has [1]

$$p_E^{(Bayes)} = \frac{1}{2}(1 - \|a_1\rho_1 - a_2\rho_2\|_1), \qquad (2)$$

where $\|A\|_1$ denotes the trace norm of A.

In the minimax problem, one does not have *a priori* probabilities. However, one defines the error probability $\varepsilon_i(\vec{P}) = \text{Tr}[\rho_i(I - P_i)]$ of failing to identify ρ_i. The optimal minimax solution consists in finding the POVM that achieves the minimax

$$\varepsilon = \min_{\vec{P}} \max_{i=1,2} \varepsilon_i(\vec{P}), \qquad (3)$$

or equivalently, that maximizes the smallest of the probabilities of correct detection

$$1 - \varepsilon = \max_{\vec{P}} \min_{i=1,2}[1 - \varepsilon_i(\vec{P})] = \max_{\vec{P}} \min_{i=1,2} \text{Tr}(\rho_i P_i). \qquad (4)$$

The minimax and Bayesian strategies of discrimination are connected by the following theorem.

Theorem 1. If there is an *a priori* probability $\vec{a} = (a_1, a_2)$ for the states ρ_1 and ρ_2, and a measurement $\vec{B}$ that achieves the optimal Bayesian average error for $\vec{a}$, with equal probabilities of correct detection, i.e.,

$$\text{Tr}(\rho_1 B_1) = \text{Tr}(\rho_2 B_2), \qquad (5)$$

then $\vec{B}$ is also the solution of the minimax discrimination problem.

Proof. In fact, suppose on the contrary that there exists a POVM $\vec{P}$ such that $\min_{i=1,2}\text{Tr}(\rho_i P_i) > \min_{i=1,2}\text{Tr}(\rho_i B_i)$. Due to assumption (5) one has $\text{Tr}(\rho_i P_i) > \text{Tr}(\rho_i B_i)$ for both $i = 1, 2$, whence

$$\sum_i a_i \text{Tr}(\rho_i P_i) > \sum_i a_i \text{Tr}(\rho_i B_i) \qquad (6)$$

which contradicts the fact that $\vec{B}$ is optimal for $\vec{a}$. ∎

The existence of an optimal $\vec{B}$ as in Theorem 1 will be shown in the following.

First, by labeling with $\vec{P}^{(a)}$ an optimal POVM for the Bayesian problem with prior probability distribution $\vec{a} = (a, 1-a)$, and defining

$$\chi(a, \vec{P}) \doteq a\text{Tr}(\rho_1 P_1) + (1-a)\text{Tr}(\rho_2 P_2), \qquad (7)$$

we have the following lemma.

Lemma 1. The function $f(a) \doteq \text{Tr}(\rho_1 P_1^{(a)}) - \text{Tr}(\rho_2 P_2^{(a)})$ is monotonically nondecreasing, with minimum value $f(0) \leq 0$, and maximum value $f(1) \geq 0$.

In fact, consider $\vec{P}^{(a)}$ and $\vec{P}^{(b)}$ for two values a and b with $a < b$ and define $\vec{D} = \vec{P}^{(b)} - \vec{P}^{(a)}$. Then

$$\chi(a, \vec{P}^{(b)}) = \chi(a, \vec{P}^{(a)}) + \chi(a, \vec{D}),$$

$$\chi(b, \vec{P}^{(a)}) = \chi(b, \vec{P}^{(b)}) - \chi(b, \vec{D}). \qquad (8)$$

Now, since $\chi(a, \vec{P}^{(a)})$ is the optimal probability of correct detection for prior a, and analogously $\chi(b, \vec{P}^{(b)})$ for prior b, then $\chi(a, \vec{D}) \leq 0$ and $\chi(b, \vec{D}) \geq 0$, and hence

$$0 \leq \chi(b, \vec{D}) - \chi(a, \vec{D}) = (b - a)[\text{Tr}(\rho_1 D_1) - \text{Tr}(\rho_2 D_2)].$$

It follows that $\text{Tr}(\rho_1 D_1) \geq \text{Tr}(\rho_2 D_2)$, namely,

$$\text{Tr}(\rho_1 P_1^{(b)}) - \text{Tr}(\rho_1 P_1^{(a)}) \geq \text{Tr}(\rho_2 P_2^{(b)}) - \text{Tr}(\rho_2 P_2^{(a)}) \qquad (9)$$

or, equivalently,

$$\text{Tr}(\rho_1 P_1^{(b)}) - \text{Tr}(\rho_2 P_2^{(b)}) \geq \text{Tr}(\rho_1 P_1^{(a)}) - \text{Tr}(\rho_2 P_2^{(a)}). \qquad (10)$$

Equation (10) states that the function $f(a)$ is monotonically nondecreasing. Moreover, for $a = 0$ the POVM detects only the state ρ_2, whence $\text{Tr}(\rho_2 P_2^{(0)}) = 1$, and one has $f(0) = -1 + \text{Tr}[\rho_1 P_1^{(0)}] \leq 0$. Similarly one can see that $f(1) \geq 0$. ∎

We can now prove the following theorem.

Theorem 2. An optimal $\vec{B}$ as in Theorem 1 always exists.

Proof. Consider the value a_0 of a where $f(a)$ changes its sign from negative to positive, and there take the left and right limits

$$\vec{P}^{(\mp)} = \lim_{a \to a_0^{\mp}} \vec{P}^{(a)}. \qquad (11)$$

For $f(a_0^+) = f(a_0^-) = 0$ just define $\vec{B} = \vec{P}^{(a_0)}$. For $f(a_0^+) > f(a_0^-)$ define the POVM $\vec{B}$

$$\vec{B} = \frac{f(a_0^+)\vec{P}^{(-)} - f(a_0^-)\vec{P}^{(+)}}{f(a_0^+) - f(a_0^-)}. \qquad (12)$$

In fact, one has

$$\text{Tr}(\rho_1 B_1) - \text{Tr}(\rho_2 B_2) = [f(a_0^+) - f(a_0^-)]^{-1}$$
$$\times [\text{Tr}(\rho_1 P_1^{(-)} - \rho_2 P_2^{(-)})f(a_0^+)$$
$$- \text{Tr}(\rho_1 P_1^{(+)} - \rho_2 P_2^{(+)})f(a_0^-)] = 0, \qquad (13)$$

namely, Eq. (5) holds. ∎

Notice that the value a_0 is generally not unique, since the function $f(a)$ can be locally constant. However, on the Hilbert space $\text{Supp}(\rho_1) \cup \text{Supp}(\rho_2)$, the optimal POVM for the minimax problem is unique, apart from the very degenerate case in which $D = a_0\rho_1 - (1-a_0)\rho_2$ has at least a two-dimensional kernel. In fact, upon denoting by Π_+ and K the projector on the strictly positive part and the kernel of D, respectively, any Bayes optimal POVM is written $(B_1 = \Pi_+ + K', B_2 = I - B_1)$, with $K' \leq K$. Since for the optimal minimax POVM we need $\text{Tr}(\rho_1 B_1) = \text{Tr}(\rho_2 B_2)$, one obtains $\text{Tr}[(\rho_1 + \rho_2)K'] = 1 - \text{Tr}[(\rho_1 + \rho_2)\Pi_+]$, which has a unique solution $K' = \alpha K$ if K is a one-dimensional projector.

Remark 1. For two pure states the optimal POVM for the minimax discrimination is orthogonal and unique (up to trivial completion of $\text{Span}\{|\psi_i\rangle\}_{i=1,2}$ to the full Hilbert space of the quantum system).

In fact, on the space $\text{Span}\{|\psi_i\rangle\}_{i=1,2}$ the optimal Bayes measurement is always orthogonal and unique for any prior

probability; hence there exists an optimal POVM for the minimax discrimination that coincides with the optimal Bayesian one, which is orthogonal. Uniqueness of the minimax optimal POVM follows from the considerations after the proof of Theorem 2 when restricted to the subspace spanned by the two states.

Remark 2. There are couples of mixed states for which the optimal minimax POVM is unique and nonorthogonal.

For example, consider the following states in dimension 2:

$$\rho_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad \rho_2 = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{bmatrix}. \tag{14}$$

Then an optimal minimax POVM is given by

$$P_1 = \begin{bmatrix} \frac{2}{3} & 0 \\ 0 & 0 \end{bmatrix}, \quad P_2 = \begin{bmatrix} \frac{1}{3} & 0 \\ 0 & 1 \end{bmatrix}. \tag{15}$$

In fact, clearly there is an optimal POVM of the diagonal form. We need to maximize $\min_{i=1,2}\mathrm{Tr}(\rho_i P_i)$, whence, according to Theorem 2, we need to maximize $\mathrm{Tr}(\rho_1 P_1)$ with the constraints $\mathrm{Tr}(\rho_1 P_1)=\mathrm{Tr}(\rho_2 P_2)$ and $P_2 = I - P_1$. Such an optimal POVM is unique, otherwise there would exist a convex combination $a_0\rho_1 - (1-a_0)\rho_2$ with kernel at least two dimensional, which is impossible in the present example (see comments after the proof of Theorem 2).

Notice that when the optimal POVM for the minimax strategy is unique and nonorthogonal, then there is a prior probability distribution $\vec{a}$ for which the optimal POVM for the Bayes problem is not unique, and the nonorthogonal POVM that optimizes the minimax problem is also optimal for the Bayes one. In the example of Remark 2 the optimal POVM (15) is also optimal for the Bayes problem with $\vec{a} = \left(\frac{1}{3}, \frac{2}{3}\right)$ as one can easily check. However, in the Bayes case one can always choose an optimal orthogonal POVM, whereas in the minimax case you may have to choose a nonorthogonal POVM.

Finally, notice that, unlike in the Bayesian case, the optimal POVM for the minimax strategy may also be not extremal.

III. OPTIMAL MINIMAX DISCRIMINATION OF $N \geq 2$ QUANTUM STATES

We now consider the easiest case of discrimination with more than two states, namely, the discrimination among a covariant set. In a fully covariant state discrimination, one has a set of states $\{\rho_i\}$ with $\rho_i = U_i \rho_0 U_i^\dagger \; \forall \; i$, for fixed ρ_0 and $\{U_i\}$ a (projective) unitary representation of a group. In the Bayesian case full covariance requires that the prior probability distribution $\{a_i\}$ is uniform. Then one can easily prove (see, for example, Ref. [8]) that also the optimal POVM is covariant, namely, it is of the form $P_i = U_i K U_i^\dagger$, for suitable fixed operator $K \geq 0$.

Theorem 3. For a fully covariant state discrimination problem, there is an optimal measurement for the minimax strategy that is covariant, and coincides with an optimal Bayesian measurement.

Proof. A covariant POVM $\{P_i\}$ gives a probability $p = \mathrm{Tr}(\rho_i P_i)$ independent of i. Moreover, there always exists an optimal Bayesian POVM that is covariant and maximizes p, which then is also the maximum over all POVM's of the average probability of correct estimation $\overline{\mathrm{Tr}(\rho_i P_i)}$ for uniform prior distribution [8]. Now, suppose by contradiction that there exists an optimal minimax POVM $\{P_i'\}$ maximizing $p' = \underline{\min_i \mathrm{Tr}(\rho_i P_i')}$, for which $p' > p$. Then, one has $p < p' \leq \overline{\mathrm{Tr}(\rho_i P_i')}$, contradicting the assertion that an optimal Bayesian POVM maximizes $\overline{\mathrm{Tr}(\rho_i P_i)}$ over all POVM's. Therefore, $p = p'$, and the covariant Bayesian POVM also solves the minimax problem. ∎

Notice that in the covariant case also for any optimal minimax POVM $\{P_i\}$ one has $\mathrm{Tr}(\rho_i P_i)$ independent of i, since the average probability of correct estimation is equal to the minimum one.

In the following we generalize Theorem 1 for two states to the case of $N \geq 2$ states and arbitrary weights. We have the following theorem.

Theorem 4. For any set of states $\{\rho_i\}_{2 \leq i \leq N}$ and any set of weights w_{ij} (price of misidentifying i with j) the solution of the minimax problem

$$r = \inf_{\vec{P}} \sup_i \sum_j w_{ij} \mathrm{Tr}(\rho_i P_j) \tag{16}$$

is equivalent to the solution of the problem

$$r = \max_{\vec{a}} r_B(a), \tag{17}$$

where $r_B(\vec{a})$ is the Bayesian risk

$$r_B(\vec{a}) \doteq \max_{\vec{P}} \sum_i a_i \sum_j w_{ij} \mathrm{Tr}(\rho_i P_j). \tag{18}$$

Proof. The minimax problem in Eq. (16) is equivalent to looking for the minimum of the real function $\delta = f(\vec{P})$ over $\vec{P}$, with the constraints

$$\sum_j w_{ij} \mathrm{Tr}(\rho_i P_j) \leq \delta, \quad \forall \; i,$$

$$P_j \geq 0, \quad \forall \; j,$$

$$\sum_j P_j = I. \tag{19}$$

Upon introducing the Lagrange multipliers

$$\mu_i \in \mathbb{R}^+, \quad \forall \; i,$$

$$0 \leq Z_i \in M_d(\mathbb{C}), \quad \forall \; i,$$

$$Y^\dagger = Y \in M_d(\mathbb{C}), \tag{20}$$

$M_d(\mathbb{C})$ denoting the $d \times d$ matrices on the complex field, the problem is equivalent to

$$r = \inf_{\vec{P},\delta} \sup_{\vec{\mu},\vec{Z},Y}' l(\vec{P}, \delta, \vec{\mu}, \vec{Z}, Y),$$

$$l(\vec{P},\delta,\vec{\mu},\vec{Z},Y) \doteq \delta + \sum_i \left[\mu_i \left(\sum_j w_{ij}\mathrm{Tr}(\rho_i P_j) - \delta \right) \right]$$
$$- \sum_i \mathrm{Tr}(Z_i P_i) + \mathrm{Tr}\left[Y\left(I - \sum_i P_i \right) \right],$$

(21)

where sup$'$ denotes the supremum over the set defined in Eqs. (20). The problem is convex [namely, both the function δ and the constraints (19) are convex] and meets Slater's conditions [9] (namely, one can find values of $\vec{P}$ and δ such that the constraints are satisfied with strict inequalities), and hence in Eq. (21) one has

$$\inf_{\vec{P},\delta} \sup'_{\vec{\mu},\vec{Z},Y} l(\vec{P},\delta,\vec{\mu},\vec{Z},Y) = \max'_{\vec{\mu},\vec{Z},Y} \inf_{\vec{P},\delta} l(\vec{P},\delta,\vec{\mu},\vec{Z},Y). \quad (22)$$

It follows that

$$r = \max'_{\vec{\mu},\vec{Z},Y} \mathrm{Tr}Y \qquad (23)$$

under the additional constraints

$$\sum_i \mu_i = 1,$$

$$\sum_i w_{ij}\mu_i\rho_i - Z_j - Y = 0, \qquad \forall\ j. \qquad (24)$$

The constraints can be rewritten as

$$\mu_i \geqslant 0, \qquad \sum_i \mu_i = 1,$$

$$Y \leqslant \sum_i w_{ij}\mu_i\rho_i, \qquad \forall\ j. \qquad (25)$$

Now, notice that for the Bayesian problem with prior $\vec{a}$, along the same reasoning, one writes the equivalent problem

$$r_B(\vec{a}) = \max'_{Y} \mathrm{Tr}Y, \qquad (26)$$

with the constraint

$$\sum_i w_{ij}a_i\rho_i - Z_j - Y = 0, \qquad \forall\ j, \qquad (27)$$

$$a_i \geqslant 0, \qquad \sum_i a_i = 1,$$

$$Y \leqslant \sum_i w_{ij}a_i\rho_i, \qquad \forall\ j. \qquad (28)$$

which is the same as the minimax problem, with the role of the Lagrange multipliers $\{\mu_i\}$ now played by the prior probability distribution $\{a_i\}$. ∎

Clearly, a POVM that attains r in the minimax problem (16) actually exists, being the infimum over a (weakly) compact set—the POVM convex set—of the (weakly) continuous function $\sup_i \Sigma_j w_{ij}\mathrm{Tr}(\rho_i P_j)$.

IV. OPTIMAL MINIMAX UNAMBIGUOUS DISCRIMINATION

In this section we consider the so-called unambiguous discrimination of states [4], namely, with no error, but possibly with an inconclusive outcome of the measurement. We focus attention on a set of N pure states $\{\psi_i\}_{i \in \mathsf{S}}$. In such a case, it is possible to have unambiguous discrimination only if the states of the set S are linearly independent, whence there exists a biorthogonal set of vectors $\{|\omega_i\rangle\}_{i \in \mathsf{S}}$, with $\langle\omega_i|\psi_j\rangle = \delta_{ij}$, $\forall\ i,j \in \mathsf{S}$. We will conveniently restrict our attention to $\mathrm{Span}\{|\psi_i\rangle\}_{i \in \mathsf{S}} \equiv \mathsf{H}$ (otherwise one can trivially complete the optimal POVM for the subspace as a POVM for the full Hilbert space of the quantum system). While in the Bayes problem the probability of inconclusive outcome is minimized, in the minimax unambiguous discrimination we need to maximize $\min_i\langle\psi_i|P_i|\psi_i\rangle$ over the set of POVM's with $\langle\psi_i|P_j|\psi_i\rangle = 0$ for $i \neq j \in \mathsf{S}$, and the POVM element that pertains to the inconclusive outcome will be given by $P_{N+1} = I - \Sigma_{i \in \mathsf{S}}P_i$. We have the following theorem.

Theorem 5. The optimal minimax unambiguous discrimination of N pure states $\{\psi_i\}_{i \in \mathsf{S}}$ is achieved by the POVM

$$P_i = \kappa|\omega_i\rangle\langle\omega_i|, \quad i \in \mathsf{S},$$

$$P_{N+1} = I - \sum_{i \in \mathsf{S}} P_i, \qquad (29)$$

where κ is given by

$$\kappa^{-1} = \text{max eigenvalue of } \sum_{i \in \mathsf{S}} |\omega_i\rangle\langle\omega_i|. \qquad (30)$$

Proof. We need to maximize $\min_i\langle\psi_i|P_i|\psi_i\rangle$ over the set of POVM's with $\langle\psi_i|P_j|\psi_i\rangle = 0$ for $i \neq j \in \mathsf{S}$, whence clearly $P_j = \kappa_j|\omega_j\rangle\langle\omega_j|$. Then the problem is to maximize $\min_{i \in \mathsf{S}} \kappa_i$. This can be obtained by taking $\kappa_i = \kappa$ independent of i and then maximizing κ. In fact, if there is a $\kappa_i > \kappa_j$ for some i,j, then we can replace κ_i with κ_j, and iteratively we get $\kappa_i = \kappa$ independently of i. Finally, the maximum κ giving $P_{N+1} \geqslant 0$ is the one given in the statement of the theorem. ∎

As regards the unicity of the optimal POVM, we can show the following.

Theorem 6. The optimal POVM of Theorem 5 is nonunique if and only if $|\omega_i\rangle \in \mathrm{Supp}(P_{N+1})$ for some $i \in \mathsf{S}$.

Proof. In fact, if there exists an $i \in \mathsf{S}$ such that $|\omega_i\rangle \in \mathrm{Supp}(P_{N+1})$, this means that there exists $\varepsilon > 0$ such that $\varepsilon|\omega_i\rangle\langle\omega_i| \leqslant P_{N+1}$. Then the following is a POVM:

$$Q_j = P_j \quad \text{for } j \neq i,$$

$$Q_i = P_i + \varepsilon|\omega_i\rangle\langle\omega_i|,$$

$$Q_{N+1} = P_{N+1} - \varepsilon|\omega_i\rangle\langle\omega_i|, \qquad (31)$$

and is optimal as well. Conversely, if there exists another equivalently optimal POVM $\{Q_j\}$, then there exists an $i \in \mathsf{S}$ such that $Q_i > P_i$ (since both are proportional to $|\omega_i\rangle\langle\omega_i|$, and $\min_i\langle\psi_i|Q_i|\psi_i\rangle$ has to be maximized). Then $|\omega_i\rangle \in \mathrm{Supp}(P_{N+1})$. ∎

When the optimal POVM according to Theorem 6 is not unique, one can refine the optimality criterion in the following way. Define the set $\mathsf{S}_1 \subset \mathsf{S}$ for which one has $|\omega_i\rangle \in \mathrm{Supp}(P_{N+1})$. Denote by $\mathfrak{B}_1$ the set of POVM's that are equivalently optimal to those of Theorem 5. Then define the set of POVM's $\mathfrak{B}_2 \subset \mathfrak{B}_1$ that maximizes $\min_{i \in \mathsf{S}_1} \langle \omega_i | P_i | \omega_i \rangle$. In this way one iteratively reach a unique optimal POVM, which is just the one given in Eqs. (29) and (30).

V. CONCLUSIONS

In conclusion, we have considered the problem of optimal discrimination of quantum states in the minimax strategy. This corresponds to maximizing the smallest of the probabilities of correct detection over all measurement schemes. We have derived the optimal measurement both in the minimal-error and in the unambiguous discrimination problem for any number of quantum states. The relation between the optimal measurement and the optimal Bayesian solutions has been given. Differently from the Bayesian scenario, we have shown that there are instances in which the minimum risk cannot be achieved by an orthogonal measurement. Finally, in the unambiguous discrimination problem, we have shown a refinement of the minimax problem that leads always to a unique optimal minimax measurement.

ACKNOWLEDGMENTS

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