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MEASURING THE QUANTUM STATE OF LIGHT[1]

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A short overview is given on recently developed techniques for reconstructing the quantum state of light from data measured by means of balanced homodyne detection. In particular, a reconstruction scheme for the density operator in Fock representation is described which works also in case of inefficient detectors.

1. Introduction

The usual way of gaining information on a quantum system is to measure an appropriate observable. Clearly, such a procedure shows up only one facet of the micro-object. To get more information, one has to measure a second observable, preferably noncommuting with the first one, and so on. However, this will become a cumbersome task since, in general, one needs a completely different experimental setup for every new type of measurements. Moreover, the question arises as to whether the information gathered in this way is complete. On the other hand, it is well known that the full quantum mechanical information on a system is contained in the wave function or, more generally, the density operator. Hence it appears desirable to determine the latter from experiments, so to say to measure the quantum state. In what follows, we will give a short overview on some recent achievements in this field.

2. Reconstruction of the wave function

The problem of reconstructing Schrödinger's wave function from measured data, namely the probability distributions for both position and momentum, was addressed

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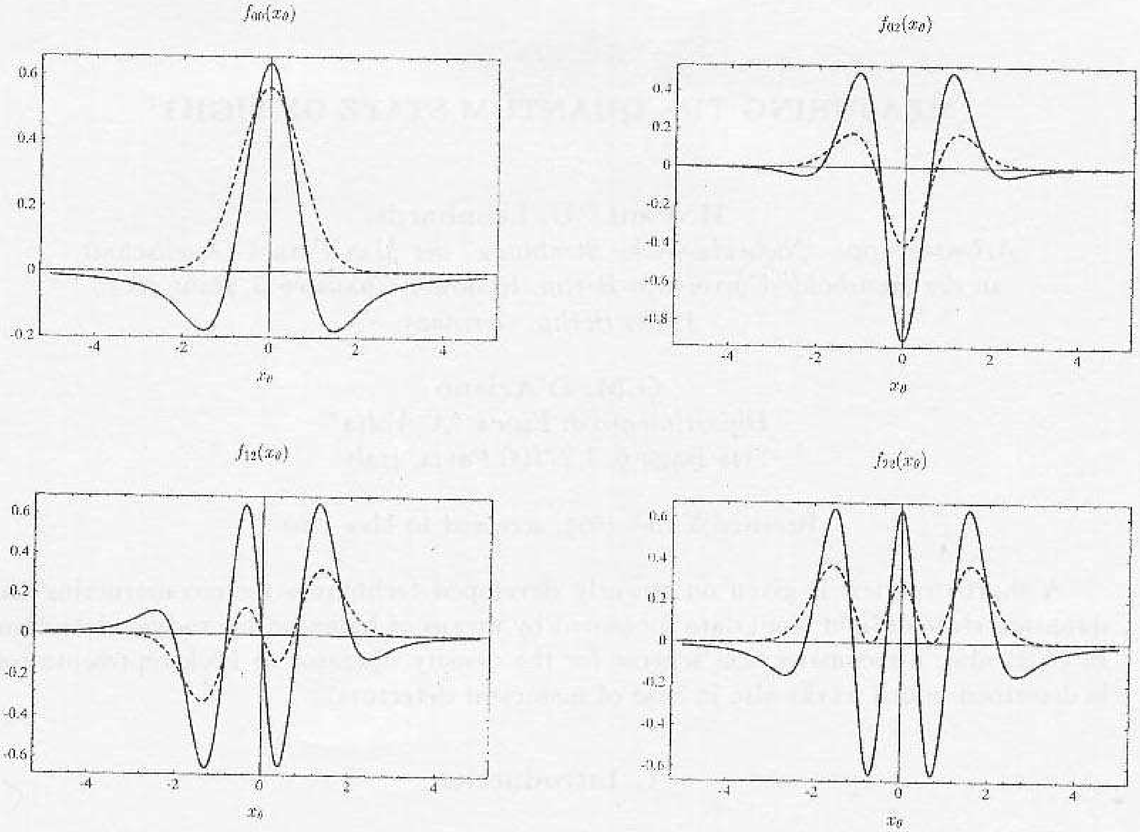

Fig. 1. We plot the x_Θ-dependent part $f_{mn}(x_\Theta)$ of the pattern function $F_{mn}(x_\Theta, \Theta) = \exp\{i(m-n)\Theta\} f_{mn}(x_\Theta)$ for different values of m, n (full line) and compare it with the corresponding product of the Schrödinger wave functions $\psi_m(x_\Theta)$ and $\psi_n(x_\Theta)$ for the Fock states $|m\rangle$ and $|n\rangle$ (broken line).

already in the early days of quantum mechanics. In his famous handbook article [1] Pauli, however, devoted only a footnote to this issue, raising only the question as to whether the wave function is determined unambiguously in this way. Actually, until now no satisfactory scheme to solve this reconstruction problem has been developed. Achievements were made, however, in the field of optical or electron microscopy [2, 3]. In fact, the mathematical task is here the same as in quantum theory: The field distribution plays the role of Schrödinger's wave function, and what can be measured is the intensity in both the image and the focal plane. Since the field distribution in the latter is nothing but the Fourier transform of the distribution in the image plane, the two intensity distributions correspond precisely to the quantum mechanical position and momentum distributions. We will not go into details of the reconstruction problem which, in fact, amounts to retrieve the phase of the optical field (or Schrödinger's wave function). We want only to point out the mathematical difficulties one encounters.

The procedure will be to expand the wave function $\psi(x)$ with respect to a suitable

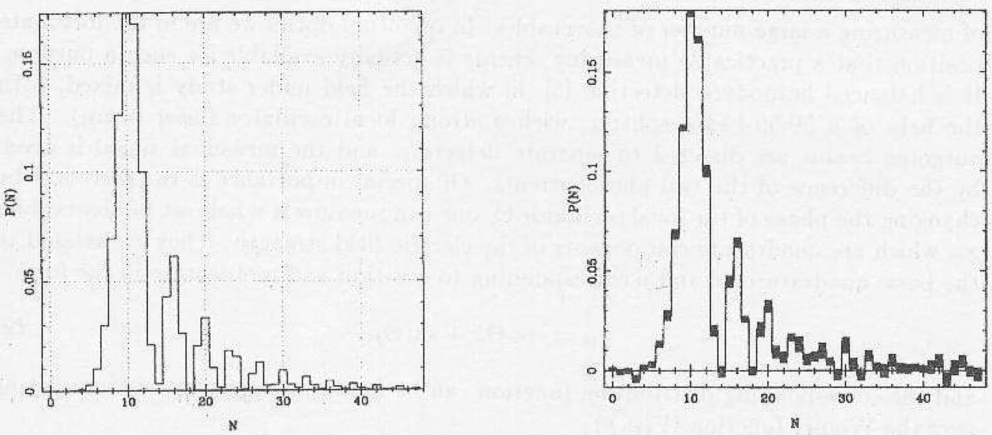

Fig. 2. We present an example for the reconstruction scheme based on Eqs.(8) and (9). Fig. 2a shows the true photon-number distribution for a squeezed state with squeezing parameter $s = 20$ and displacement parameter $\alpha = 3$, while Fig. 2b displays the distribution reconstructed from computer experiments using 260 quadrature phases with 100 simulated measurements for each phase. The errors indicated were obtained from 10 simulated runs.

orthogonal basis $u_n(x)$

$$v(x) = \sum_n c_n u_n(x) . \qquad (1)$$

In quantum optics, it appears natural to choose the u_n's as the energy eigenstates of the harmonic oscillator, i.e. Fock states in x representation [4]. Expressing the probability distributions for x and p, assumed to be given, through the unknown coefficients c_n in Eq.(1), we arrive at an infinite system of equations that are quadratic in c_n and c_n^* and, hence, cannot be solved in a simple way. An approximative scheme can be based on truncation [4]: Assuming that the coefficients c_n vanish approximately for $n > m$ (fixed), one gets just one equation for $|c_m|^2$. Since an overall phase of the wave function has no physical relevance, this gives us c_m. Then we can select two equations which, after insertion of c_m, become linear in c_{m-1} and c_{m-1}^*. They are readily solved to yield c_{m-1}. In the next step, we can select four equations which, after insertion of c_{m-1}, give us c_{m-2} and c_{m-3}. Proceeding similarly further, we find all coefficients c_n. The scheme in question is, however, unsatisfactory, since we start from a rather small coefficient c_m whose error will propagate.

Noticing that the equations we have to solve are, in fact, linear in the density matrix elements $\varrho_{mn} = c_m c_n^*$, we might say, the dilemma is that we have not enough equations to determine the ϱ_{mn}'s. Actually, they are not independent since we are dealing with a wave function, i.e. a pure state.

Hence we can expect that the reconstruction problem becomes, in fact, much simpler, from the mathematical point of view, when we study statistical mixtures which amounts to determine the density matrix elements. However, this can be done only at the expense of measuring a large number of observables. In quantum optics we are in the fortunate position that a practicable measuring scheme is actually available for such a purpose. It is balanced homodyne detection [5], in which the field under study is mixed, with the help of a 50:50 beam splitter, with a strong local oscillator (laser beam). The outgoing beams are directed to separate detectors, and the measured signal is given by the difference of the two photocurrents. Of special importance is the fact that by changing the phase of the local oscillator Θ, one can measure a whole set of observables x_Θ which are quadrature components of the electric field strength. They are related to the basic quadratures x and p corresponding to position and momentum in the form

$$x_\Theta = \cos\Theta x + \sin\Theta p\,, \tag{2}$$

and the corresponding distribution function can be found by appropriately integrating over the Wigner function $W(x,p)$

$$w_\Theta(x_\Theta) = \int_{-\infty}^{\infty} W(x_\Theta\cos\Theta - p\sin\Theta,\, x_\Theta\sin\Theta + p\cos\Theta)dp\,. \tag{3}$$

This relation, known as Radon transformation, is, in fact, a straight-forward generalization of the well known representations of the position and momentum distributions as marginals of the Wigner function.

3. Optical homodyne tomography

The reconstruction of the Wigner function from the distributions (3) is formally identical to what is done in medical tomography, and hence has been called optical homodyne tomography [6]. Actually, in a pioneering work, Vogel and Risken [7] solved this problem more generally, namely, they derived a reconstruction formula for the so-called s-parametrized quasiprobability functions introduced by Cahill and Glauber [8]

$$W(x,p;s) = \frac{1}{(2\pi)^2}\int_{-\infty}^{\infty} d\xi \int_0^{\pi} d\Theta \int_{-\infty}^{\infty} dx_\Theta e^{s\xi^2/4 + i\xi(x_\Theta - x\cos\Theta - p\sin\Theta)}|\xi| w_\Theta(x_\Theta)\,. \tag{4}$$

From convergency requirements the parameter s has to be restricted to $s \leq 0$. The value $s = 0$ corresponds to the Wigner function. With the help of the filtered back projection algorithm elaborated in medical tomography, Raymer's group [6] succeeded in reconstructing the Wigner function for some relevant cases.

It should be emphasized that the Wigner function actually comprises the full quantum mechanical information on the system. By means of a simple Fourier transformation one obtains the density operator in x representation from which it is easy to pass, in particular, to the Fock representation. The density operator being at hand, one can readily calculate the distribution function for any observable, even when it is defined only as a POM (probability operator measure) such as the quantum phase of light [6].

4. Direct determination of the density matrix

As a matter of fact, the procedure mentioned before is rather involved, and, moreover, the basic step, the reconstruction of the Wigner function (inverse Radon transformation) needs filtering which, together with discrete sampling with respect to Θ, removes any high-frequency components being eventually present in the true Wigner function. From the mathematical point of view, the situation is more favourable for $s < 0$, since then the integral (4) contains a convergency producing Gaussian. In this context one should remember that the density operator, in Fock representation, can be obtained by differentiation from the Q function corresponding to $s = -1$. Hence the following strategy recommends itself: Start from the closed-form expression for the Q function given by Eq.(4) for $s = -1$ and perform the appropriate differentiations on it before carrying out the integration over the measured distribution functions $w_\Theta(x_\Theta)$.

The first to adopt this procedure were D'Ariano et al. [9] who started from the well known relation

$$\varrho_{mn} = (m!n!)^{-\frac{1}{2}} \frac{\partial^m}{\partial \overline{\alpha}^m} \frac{\partial^n}{\partial \alpha^n} \left(e^{|\alpha|^2} Q(\alpha, \overline{\alpha}) \right) \Big|_{\alpha=\overline{\alpha}=0} , \tag{5}$$

where $\alpha = 2^{-\frac{1}{2}}(x + ip)$. However, they obtained rather involved formulas for the matrix elements ϱ_{mn}. The formalism becomes, in fact, more transparent when it is based on Kano's formula [10]

$$\varrho_{n+\lambda,n} = \frac{1}{2} \sqrt{\frac{n!}{(n+\lambda)!}} \frac{d^{n+\lambda}}{dy^{n+\lambda}} F_\lambda(y) \Big|_{y=0} \quad (\lambda \geq 0) , \tag{6}$$

where

$$F_\lambda(y) = e^y \int_0^{2\pi} Q(\alpha) \alpha^{-\lambda} d\delta, \quad \alpha = \sqrt{y} e^{i\delta} . \tag{7}$$

Then one arrives at the following result [11]

$$\varrho_{n+\lambda,n} = \int_0^\pi d\Theta \int_{-\infty}^{\infty} dx_\Theta F(n+\lambda, n; x_\Theta, \Theta) w_\Theta(x_\Theta) . \tag{8}$$

Here, the "pattern function" F is given by

$$\begin{aligned} F(n+\lambda, n; x_\Theta, \Theta) \quad = \quad & e^{i\lambda\Theta} \frac{1}{\pi} \sqrt{\frac{n!}{(n+\lambda)!}} \sum_{\nu=0}^{n} b_\nu^{(n,\lambda)} 2^{\nu+1} (2\nu + \lambda + 1)! e^{-x_\Theta^2/2} \\ & \times \mathrm{Re} \left\{ (-i)^\lambda D_{-[2\nu+\lambda+2]} \left(-i 2^{\frac{1}{2}} x_\Theta \right) \right\} , \end{aligned} \tag{9}$$

where

$$b_\nu^{(n,\lambda)} = (-1)^\nu (2^\nu \nu!)^{-1} \binom{n+\nu}{n-\nu} \tag{10}$$

and $D_k(z)$ denotes the parabolic cylinder function. The pattern function has the asymptotic behaviour

$$F(n+\lambda; n; x_\Theta, \Theta) = \text{const } e^{i\lambda\Theta} x_\Theta^{-(\lambda+2)} \quad (\lambda \geq 0) , \tag{11}$$

i.e., it goes to zero as an inverse power of x_Θ. This feature is of practical relevance, since it provides an effective cut-off for the measured distributions. It is interesting to note that the pattern functions resemble closely the corresponding product of the Schrödinger wave functions $\psi_n(x_\Theta)$ and $\psi_m(x_\Theta)$ for the Fock states $|n\rangle$ and $|m\rangle$, as can be seen from Fig. 1.

Simulating a large number of measurements, the photon distribution for a squeezed state has been reconstructed with the help of the basic relation (8). As becomes obvious from Fig. 2, the scheme works very well.

5. Inefficient detectors

It is well known that inefficient detectors deteriorate the measurements. When reconstructing the Wigner function in the tomographic scheme, the effect of nonunit detection efficiency η is correctly accounted for [12] by stating that what is actually reconstructed is not the true Wigner function $W(x,p)$ but the s-parametrized quasiprobability distribution $\eta^{-1}W(2^{-\frac{1}{2}}x, 2^{-\frac{1}{2}}p; (\eta - 1/\eta)$. The latter is actually a smoothed version of the Wigner function, as becomes obvious from the general relationship between s-parametrized quasiprobability functions [8]

$$W(x,p;s) = \pi^{-1}(t-s)^{-1}\int_{-\infty}^{\infty} dx' \int_{-\infty}^{\infty} dp' W(x',p';t) \times \exp\left\{-(t-s)^{-1}[(x-x')^2 + (p-p')^2]\right\}. \qquad (12)$$

This formula allows one to determine, in particular, any distribution with $s < 0$ from the Wigner function by convolution with a simple Gaussian. We can utilize Eq.(12) to pass from the above mentioned quasiprobability distribution reconstructed with the help of inefficient detectors to the Q function. We thus arrive at the result

$$\begin{aligned} Q(x,p) &\equiv W(x,p;-1) \\ &= \eta(2\pi)^{-2}\int_{-\infty}^{\infty} d\xi \int_0^{\pi} d\Theta \int_{-\infty}^{\infty} dx_\Theta \\ &\quad \times \exp\left\{-(2\eta-1)\xi^2/4 + \mathrm{i}\xi(x_\Theta - \eta^{\frac{1}{2}}x\cos\Theta - \eta^{\frac{1}{2}}p\sin\Theta)\right\} \\ &\quad \times|\xi| w_\Theta^{(\eta)}(x_\Theta) \end{aligned} \qquad (13)$$

which is, in fact, the extension of the Vogel-Risken formula (4) for $s = -1$ to the case of inefficient detectors. Now one can proceed as before, namely determine from the Q function, using Kano's formula (6), the density operator in Fock representation.

One learns from Eq.(13) that there is a critical lower limit to the detector efficiency η: It has to be larger than $1/2$, since otherwise an inverse Gaussian producing divergency, will occur in the integrand. When this condition is fulfilled, the reconstruction scheme has been shown by simulations to work pretty well, however, much higher accuracy is needed, compared to the ideal case $\eta = 1$, which means that the number of measurements has to be drastically enhanced [13]. This is the price one has to pay for effectively inverting the convolution originating from nonunit detection efficiency. What one learns from this is that the noise introduced by inefficient detectors does not give rise to an irretrievable loss of information, the finer details of the quantum state are only suppressed, but not destroyed.

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