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## RAPID COMMUNICATIONS

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# Two-mode heterodyne phase detection

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We present an experimental scheme that achieves ideal phase detection on a two-mode field. The two modes $a$ and $b$ are the signal and image band modes of a heterodyne detector, with the field approaching an eigenstate of the photocurrent $\hat{Z}=a+b^{\dagger}$. The field is obtained by means of a high-gain phase-insensitive amplifier followed by a high-transmissivity beam splitter with a strong local oscillator at the frequency of one of the two modes.

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The quantum-mechanical measurement of the phase of the radiation field is the essential problem of highly sensitive interferometry, and has received much attention in quantum optics [1,2]. Most of the work has been devoted to measurements on a single-mode electromagnetic field, where the measurement cannot be achieved exactly, even in principle, due to the lack of a unique self-adjoint operator [3].

It can be readily recognized that the absence of a proper self-adjoint operator in the one-mode case is mainly due to the semiboundedness of the spectrum of the number operator [4,5], which is canonically conjugated to the phase in the sense of a Fourier-transform pair [6]. This observation discloses the route toward an exact phase measurement in terms of two-mode fields, where a phase-difference operator becomes conjugated to an unbounded number-difference operator [7]. Moreover, as already noticed in Ref. [8], a two mode field corresponds to a complex photocurrent $\hat{Z}$ such that $[\hat{Z},\hat{Z}^{\dagger}]=0$, with a self-adjoint phase operator $\hat{\phi}=\arg(\hat{Z})$ that can concretely be measured. Despite its promising possibilities, not much work has been devoted to the two-mode phase detection, and attention has been focused mostly on the algebraic structure of the photocurrents (see Refs. [5–7] and references therein). Only in Ref. [8] has a concrete experimental setup been devised, based on unconventional field heterodyning with the signal and image-band modes both nonvacuum.

Here in this Rapid Communication, following the route opened by Ref. [8], we study the eigenstates of the heterodyne photocurrent $\hat{Z}$ and provide an experimental scheme that approaches them. We then analyze the measurement of the two-mode phase $\hat{\phi}=\arg(\hat{Z})$, showing that the ideal sensitivity limit $\delta\phi=1/\bar{n}$ can be achieved for a large mean number of photons $\bar{n}$.

It has been proved by Yuen and Shapiro [9] that the output photocurrent $\hat{Z}$ of a heterodyne detector (for unit quantum efficiency, and in the limit of strong local oscillator and vanishing beam splitter reflectivity) is just the operator $\hat{Z}=a+b^{\dagger}$, where $a$ denotes (the annihilator of) the signal mode and $b$ the image-band mode. In ordinary heterodyning the image-band mode $b$ is vacuum and is responsible for the additional 3-dB noise. Here, similarly to Ref. [8], we use the heterodyne detector in an unconventional way, namely with a nonvacuum $b$ mode, and look for field states that are eigenvectors of the current $\hat{Z}$.

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It is easy to check that the following vector [8],

$$|z\rangle\rangle = \int_{-\infty}^{+\infty} \frac{dx}{\sqrt{\pi}} e^{2ix\,\mathrm{Im}z} |x\rangle_0 \otimes |\mathrm{Re}z - x\rangle_{\pi/2}$$

$$= \int_{-\infty}^{+\infty} \frac{dy}{\sqrt{\pi}} e^{-2iy\,\mathrm{Re}z} |y + \mathrm{Im}z\rangle_0 \otimes |y\rangle_{\pi/2}, \qquad (1)$$

is the eigenvector of $\hat{Z}$ with complex eigenvalue $z$. In Eq. (1) $|\psi\rangle \otimes |\varphi\rangle$ denotes a vector in the two-mode Hilbert space $\mathscr{H} = \mathscr{H}_a \otimes \mathscr{H}_b$, and $|x\rangle_\phi$ represents an eigenvector of the quadrature $\hat{X}_\phi = \frac{1}{2}(c^\dagger e^{i\phi} + \mathrm{H.c.})$ of the pertaining mode $c = a,b$. The notation $|\rangle\rangle$ is a reminder that the state is a two-mode one. The set $\{|z\rangle\rangle\}$ is completely orthonormal for $\mathscr{H}$, with scalar product

$$\langle\langle z | z' \rangle\rangle = \delta^{(2)}(z - z') \equiv \delta(\mathrm{Re}z - \mathrm{Re}z')\,\delta(\mathrm{Im}z - \mathrm{Im}z'). \qquad (2)$$

In the number representation the vector (1) reads as follows:

$$|z\rangle\rangle = e^{i\,\mathrm{Re}z\,\mathrm{Im}z} \sum_{n,m=0}^{\infty} c_{n,m}(z,\bar{z}) |n\rangle \otimes |m\rangle, \qquad (3)$$

with

$$c_{n,n+\lambda}(z,\bar{z}) = \bar{c}_{n+\lambda,n}(z,\bar{z})$$

$$= \frac{(-)^n}{\sqrt{\pi}} \sqrt{\frac{n!}{(n+\lambda)!}} \bar{z}^\lambda L_n^\lambda(|z|^2) \exp\left(-\frac{1}{2}|z|^2\right). \qquad (4)$$

Equation (4) is obtained from Eq. (1), using the number representation of the quadrature

$${}_\phi\langle x | n \rangle = \left(\frac{2}{\pi}\right)^{1/4} \frac{e^{in\phi}}{\sqrt{2^n n!}} e^{-x^2} H_n(\sqrt{2}x), \qquad (5)$$

along with the following identity between Hermite and Laguerre polynomials:

$$\int_{-\infty}^{+\infty} \frac{dx}{\sqrt{\pi}} e^{-x^2} H_n(x+y) H_{n+\lambda}(x+t) = 2^{n+\lambda} n! L_n^\lambda(-2yt) t^\lambda. \qquad (6)$$

The Dirac-normalized states $|z\rangle\rangle$ have an infinite total number of photons, and we seek physically realizable states approaching $|z\rangle\rangle$ for infinite photon numbers. The eigenstate corresponding to a zero eigenvalue is given by

$$|0\rangle\rangle = \frac{1}{\sqrt{\pi}} \sum_{n=0}^{\infty} (-)^n |n\rangle \otimes |n\rangle. \qquad (7)$$

This is just the "twin beams" at the output of a phase-insensitive amplifier (PIA) in the limit of infinite gain [10]. One has

$$|0\rangle\rangle = \lim_{\lambda \to 1^-} |0\rangle\rangle_\lambda, \qquad (8)$$

with

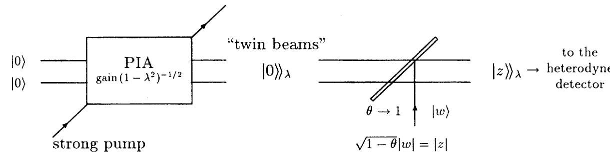


FIG. 1. Outline of the experimental setup to generate two-mode phase states approaching heterodyne eigenstates. The PIA produces the "twin beams" in Eq. (9) and the beam splitter achieves the displacement (11) (see text).

$$|0\rangle\rangle_\lambda = (1 - \lambda^2)^{1/2} \sum_{n=0}^{\infty} (-\lambda)^n |n\rangle \otimes |n\rangle$$

$$= \exp\,[\tanh^{-1}\lambda(ab - a^\dagger b^\dagger)]|0\rangle \otimes |0\rangle. \qquad (9)$$

In the parametric approximation of the infinite classical (undepleted) pump the modes $a$ and $b$ are identified with a couple of signal and idler modes of the amplifier [the gain is $(1-\lambda^2)^{-1}$]. Apart from an irrelevant phase factor, the eigenstate $|z\rangle\rangle$ can be generated by $|0\rangle\rangle$ upon displacing either $a$ or $b$. Displacing the mode $a$, we have

$$|z\rangle\rangle = e^{i\phi_z} e^{za^\dagger - \bar{z}a} |0\rangle\rangle. \qquad (10)$$

The physical (normalizable) state $|z\rangle\rangle_\lambda$ approaching $|z\rangle\rangle$ for infinite gain is obtained in the same way,

$$|z\rangle\rangle_\lambda = e^{i\phi_z} e^{za^\dagger - \bar{z}a} (1 - \lambda^2)^{1/2} \sum_{n=0}^{\infty} (-\lambda)^n |n\rangle \otimes |n\rangle. \qquad (11)$$

The displacement in Eq. (11) can be achieved by combining the "twin beams" $|0\rangle\rangle_\lambda$ with a strong coherent local oscillator $|\beta\rangle (\beta \to \infty)$ in a beam splitter with a transmissivity $\tau \to 1$, such that $|\beta|\sqrt{1-\tau} = |z|$ (the local oscillator is at the frequency of the signal mode $a$). The experimental setup to generate the state (11) is sketched in Fig. 1. The state (11) has an average number of photons

$$\bar{n} = {}_\lambda\langle\langle z | a^\dagger a + b^\dagger b | z \rangle\rangle_\lambda = |z|^2 + \frac{2\lambda^2}{1 - \lambda^2}. \qquad (12)$$

The state (11) is now impinged into a heterodyne detector with signal mode $a$ and image-band mode $b$. The probability density of getting the value $z$ for the output photocurrent $\hat{Z}$ with the field in the state $|w\rangle\rangle_\lambda$ is given by

$$|\langle\langle z | w \rangle\rangle_\lambda|^2 = (1 - \lambda^2) \left| \sum_{n=0}^{\infty} (-\lambda)^n c_{n,n}(z - w, \overline{z - w}) \right|^2$$

$$= \frac{1 - \lambda^2}{\pi} \exp(-|z - w|^2) \left| \sum_{n=0}^{\infty} \lambda^n L_n(|z - w|^2) \right|^2$$

$$= \frac{1}{\pi \Delta_\lambda^2} \exp\left(-\frac{|z - w|^2}{\Delta_\lambda^2}\right), \qquad (13)$$

where

$$\Delta_\lambda^2=\frac{1-\lambda}{1+\lambda}. \tag{14}$$

In the limit $\lambda\to 1^-$, one has that $|\langle\langle z|w\rangle\rangle_\lambda|^2\to\delta^{(2)}(z-w)$, confirming that the state $|w\rangle\rangle_\lambda$ approaches the eigenstate $|w\rangle\rangle$ of the current $\hat{Z}$.

The detection of the phase $\hat{\phi}=\arg(\hat{Z})$ is described by the marginal probability density of (13), namely,

$$\begin{aligned} p(\phi)&=\frac{1}{\pi\Delta_\lambda^2}\int_0^{+\infty}dr\ r\ \exp\left(-\frac{|re^{i\phi}-|w|e^{i\theta}|^2}{\Delta_\lambda^2}\right)\\ &=\frac{1}{2\pi}e^{-|w|^2/\Delta_\lambda^2}+\frac{|w|}{\pi\Delta_\lambda}\cos(\phi-\theta)\\ &\quad\times\frac{\sqrt{\pi}}{2}\left[1+\mathrm{erf}\left(\frac{|w|\cos(\phi-\theta)}{\Delta_\lambda}\right)\right]\\ &\quad\times e^{-(|w|^2/\Delta_\lambda^2)\sin^2(\phi-\theta)}, \end{aligned} \tag{15}$$

where $\theta=\arg(w)$, and $\mathrm{erf}(x)$ denotes the error function $\mathrm{erf}(x)=2/\sqrt{\pi}\int_0^x dte^{-t^2}$. Notice that the probability density (15) is just the Born rule for the self-adjoint operator $\hat{\phi}=\arg(\hat{Z})=-i/2\ \ln(\hat{Z}/\hat{Z}^\dagger)$; this is well defined on the Hilbert space $\mathscr{H}_0^\perp$, the orthogonal complement in $\mathscr{H}$ of the space $\mathscr{H}_0$ spanned by vector $|0\rangle\rangle$ in Eq. (7) [11]. The integral over $r$ in Eq. (15) just sums up degeneracies of eigenvectors (3); the zero-eigenvalue vector is not degenerate, and gives a zero-measure contribution to the integral. The first Gaussian term in the last line of Eq. (15) gives a uniform phase probability distribution for the "twin-beams" input state $|0\rangle\rangle_\lambda$.

For $\Delta_\lambda\ll|w|$, Eq. (15) approaches the Gaussian form

$$p(\phi)\simeq\frac{|w|}{\sqrt{\pi}\Delta_\lambda}\exp\left[-\frac{|w|^2}{\Delta_\lambda^2}(\phi-\theta)^2\right] \tag{16}$$

corresponding to the rms phase sensitivity [12]

$$\delta\phi=\langle\Delta\phi^2\rangle^{1/2}=\frac{1}{\sqrt{2}}\frac{\Delta_\lambda}{|w|}. \tag{17}$$

In the limit of infinite gain at the PIA $(\lambda\to 1^-)$ one has $\Delta_\lambda^2\simeq 1/2(1-\lambda)$ and $\bar{n}\simeq|w|^2+(1-\lambda)^{-1}$. [Notice that the classical approximation for the local oscillator at the beam splitter requires that its intensity $|\beta|^2$ be much greater than the input photon number $\simeq(1-\lambda)^{-1}$ of the "twin beams."] Optimizing $\delta\phi$ versus $|w|$ at fixed $\bar{n}$ one obtains the sensitivity

$$\delta\phi\simeq\frac{1}{\bar{n}} \tag{18}$$

for $|w|^2=(1-\lambda)^{-1}$, namely for signal photons equal to the "twin-beams" photons. The sensitivity (18) obeys the same power law as the ideal sensitivity for one-mode phase detection (actually it is improved by a constant factor equal to 1.36; see Ref. [1]).

The ideal phase sensitivity (18) has been derived with the hypothesis of unit efficiency at the heterodyne photodetector. It is easy to show that for nonunit quantum efficiency (independent of frequency in the range between signal and image-band modes) Eq. (14) becomes

$$\Delta_\lambda^2\to\Delta_\lambda^2(\eta)=\Delta_\lambda^2+\frac{1-\eta}{\eta}. \tag{19}$$

Then, it is clear that the result (18) holds only in the limit $1-\eta\ll|w|^{-2}$, whereas in the opposite situation $1-\eta\gg|w|^{-2}$ one obtains the usual shot noise $\delta\phi=\sqrt{(1-\eta)/2\bar{n}}$.

In conclusion, we have presented a feasible scheme to detect a two-mode phase of the field, approaching the eigenstates of the heterodyne current $\hat{Z}$. The state of the field is obtained by means of a high gain PIA followed by a high transmissivity beam splitter with strong local oscillator at the signal frequency. The ideal rms sensitivity $\delta\phi=1/\bar{n}$ is achieved for large photon numbers $\bar{n}\gg 1$ and for signal photons $|w|^2=\bar{n}/2$. The gain of the PIA (parametrically ideal) is tuned to the value $g=1/4\bar{n}$, and the quantum efficiency at the photodetector must be very good, namely $1-\eta\ll 2/\bar{n}$.

Hence, the two-mode phase detection could be experimentally achieved, but the technical requirements are strict: two local oscillators plus a classical pump at the PIA (all of them coherent and at different frequencies); linear amplification for high gains, with the pump still undepleted; and very good quantum efficiency. This shows how technical difficulties can rise when going from one-mode to two-mode phase detection.

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