Quantum phase amplification

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A scheme for amplifying phase shifts is presented, based on ideal photon number deamplification. For high-sensitivity measurements phase amplification sizeably reduces the bit-error rate and enhances the mutual information retrieved from the measurement. Phase-coherent states preserve their coherence under amplification, and achieve the best amplifier performance.
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I. INTRODUCTION

In quantum communications the performances of an amplifier depend on the scheme of the channel in which the device is inserted. Noticeably, the added noise is not just intrinsic of the device, but depends on the particular observable that is detected at the receiver. Thus, different kinds of preamplifier work ideally for different kinds of detection [1,2]: the phase-insensitive amplifier (PIA) is ideal for heterodyning, the phase-sensitive amplifier (PSA) for homodyning, whereas for direct detection different types of devices—the photon number amplifier (PNA) and duplicator (PND)—have been proposed [3,4].

The case of phase amplification has never been considered in the past. Indeed, for phase-reversal keying [5] a phase amplifier is of no use, because any kind of loss η that decrease the field amplitude $\alpha\to\eta^{1/2}\alpha$ does not change the average phase $\arg(\eta^{1/2}\alpha)=\arg(\alpha)$, and the $0,\pi$ phase difference cannot be further amplified. The situation, however, is very different when the information is unavoidably coded on small phase shifts, as, for example, in interferometric high-sensitivity measurements. Here, in principle, phase amplification represents a convenient strategy to improve the overall sensitivity, increasing the information retrieved in the measurement.

In this paper we show how a phase amplifier can be advantageously used for improving phase detection, achieving large reductions of the bit-error rate (BER) and sizeable enhancements of the retrieved information compared to the nonamplified scheme of small phase-shifts detection. Quite unexpectedly it turns out that the same devices—the PNA and the PND—which work ideally for direct photon detection, can also be profitably used for phase amplification.

After analyzing the quantum description of a phase amplifier in Sec. II, we show in Sec. III how an ideal number deamplifier can achieve also ideal phase amplification. Section IV is devoted to amplification of phase-coherent states, since these are the only ones that preserve coherence under phase amplification. Section V studies in some detail the amplification of small phase shifts, providing numerical results for both the ideal situation and a realistic case based on heterodyne detection of coherent states. After some remarks, in Sec. VI, on the feasibility of ideal number deamplifiers and phase-coherent state generators, Sec. VII closes the paper.

II. PHASE AMPLIFIERS

The words ''phase amplification'' can be given a precise meaning in the context of the quantum estimation theory [6]. In our case the problem is the estimation of the phase shift φ of a pure state $|\psi\rangle$ that undergoes the unitary transformation

$$|\psi\rangle\to\exp(ia^\dagger a\varphi)|\psi\rangle, \tag{1}$$

with $a^\dagger a$ denoting the number operator of the mode a of the electromagnetic field. The state $|\psi\rangle$ itself is assumed to have a well defined phase, say φ', namely, its coefficients in the number representation are of the form

$$|\psi\rangle=\sum_{n=0}^{\infty}\psi_n|n\rangle,\quad \psi_n\equiv|\psi_n|\exp(in\varphi'). \tag{2}$$

Without loss of generality, in the following, we will consider $\varphi'=0$, i.e., all ψ_n are real non-negative, such that the ideal phase probability distribution $p(\phi)=\frac{1}{2\pi}|\Sigma_{n=0}^{\infty}\psi_n e^{in\phi}|^2$ has both maximum and mean value at $\phi=0$ (averages are evaluated within the 2π periodicity, and are reduced to the $[-\pi,\pi]$ window). A phase amplifier multiplies the shift φ by a fixed gain g, independently on the state $|\psi\rangle$. In general, this can be achieved at the expense of introducing some additional noise and of partially destroying the coherence of the input state. In order to avoid biasing, the zero-phase reference of the unshifted state $|\psi\rangle$ should be retained. This corresponds to a final state that keeps non-negative matrix elements $\rho_{nm}\geqslant 0\ \forall n,m$, because this guarantees that the ideal phase probability $p(\phi)=\frac{1}{2\pi}\Sigma_{nm=0}^{\infty}\rho_{nm}e^{i(n-m)\phi}$ still has its maximum at the mean value $\phi=0$, independently on the output state. Therefore, the most general transformation $\mathcal{A}_g$ that describes phase amplification must be of the form

$$\mathcal{A}_g(e^{ia^\dagger a\varphi}|\psi\rangle\langle\psi|e^{-ia^\dagger a\varphi})=e^{ia^\dagger ag\varphi}\hat{\rho}_\psi^{(g)}e^{-ia^\dagger ag\varphi} \tag{3}$$

with the unbiasedness condition $\psi_n\geqslant 0,\forall n\Rightarrow\langle n|\hat{\rho}_\psi^{(g)}|m\rangle\geqslant 0\ \forall n,m$ [ψ_n defined in Eq. (2)].

The quantum description of an apparatus for detecting a phase shift is given by a Born's rule in the following general form:

$$p(\phi|\varphi)\ d\phi = \mathrm{Tr}[e^{ia^\dagger a\varphi}|\psi\rangle\langle\psi|e^{-ia^\dagger a\varphi}\ d\hat{\mu}(\phi)], \qquad (4)$$

where $p(\phi|\varphi)$ is the probability density of detecting a phase shift ϕ ''conditioned'' by the actual value φ, whereas $d\hat{\mu}(\phi)$ denotes the POM (probability operator-valued measure [6]) that pertains to the apparatus. When there is no *a priori* preferred phase, the conditional probability density should have the form $p(\phi|\varphi)\equiv p(\phi-\varphi)$, namely, it should depend only on the difference between the detected and the actual value of the phase shift. In this case the POM has the *covariant* form

$$d\hat{\mu}(\phi)=e^{ia^\dagger a\phi}\hat{\zeta}e^{-ia^\dagger a\phi}\frac{d\phi}{2\pi}, \qquad (5)$$

with $\hat{\zeta}$ being a fixed positive operator. For *unbiased* measurements (i.e., φ equals the averaged ϕ) all matrix elements of the operator $\hat{\zeta}$ are non-negative in the number representation [7,8]. For heterodyne or double homodyne [9] phase detection, the phase shift is retrieved from the phase of the output complex photocurrent. In this case the operator $\hat{\zeta}$ is given by [10]

$$\hat{\zeta}=\sum_{n,m=0}^{\infty}|n\rangle\langle m|\frac{\Gamma[\frac{1}{2}(n+m)+1]}{\sqrt{n!m!}}. \qquad (6)$$

On the other hand, quantum estimation theory allows optimization of the POM in Eq. (4) at a purely abstract level, providing the theoretical description of an ideal phase measurement. For covariant measurements the ideal POM has operator $\hat{\zeta}$ given by [11,12]

$$\hat{\zeta}=\sum_{n,m=0}^{\infty}|n\rangle\langle m|. \qquad (7)$$

There is, however, no known apparatus that would approach such ideal POM.

In terms of the detector's POM a phase amplifier must achieve the dual transformation of map (3), namely,

$$\mathcal{A}_g^{\vee}[d\hat{\mu}(\phi)]=\mathcal{A}_g^{\vee}(e^{ia^\dagger a\phi}\hat{\zeta}e^{-ia^\dagger a\phi})\frac{d\phi}{2\pi}$$
$$=e^{ia^\dagger ag^{-1}\phi}\hat{\zeta}_g e^{-ia^\dagger ag^{-1}\phi}\frac{d\phi}{2\pi}, \qquad (8)$$

where $\langle n|\hat{\zeta}|m\rangle\geqslant 0\ \forall n,m \Rightarrow \langle n|\hat{\zeta}_g|m\rangle\geqslant 0\ \forall n,m$, and duality is defined through the identity of traces $\mathrm{Tr}[\mathcal{A}^{\vee}(\hat{A})\hat{B}]=\mathrm{Tr}[\hat{A}\mathcal{A}(\hat{B})]$ with $\hat{A}$ bounded and $\hat{B}$ trace-class. In fact, the POM transformation (8) is the Heisenberg-picture form of the Schrödinger-picture map (3), and gives the POM of the total detector, including the preamplification. Equation (8) is the only transformation that assures that the conditional probability (4) after amplification is just a function of $g\varphi-\phi$ for every φ.

III. IDEAL NUMBER DEAMPLIFICATION AND PHASE AMPLIFICATION

We are now in position to understand how an ideal PNA can also achieve ideal phase amplification. In Ref. [13] the Hamiltonian of the ideal PNA is derived, showing that such a device is ''canonical'' for the number-phase couple—a Fourier-transform conjugated pair [14]. By ''canonical'' we denote a device analogous to the PSA, where a quadrature is amplified while the conjugated one is simultaneously deamplified . Here the PNA, when used in the inverse way as an ideal number deamplifier, works also as a phase amplifier. Since the photon number is an integer, ideal number amplification and deamplification are attained only for integer values of the gain g [15]. Ideal number deamplification and simultaneous phase amplification are described by the unitary operator [13]

$$\hat{U}_g=\sum_{\nu=0}^{g-1}\sum_{n,m=0}^{\infty}|n\rangle\langle gn+\nu|\otimes|gm+\nu\rangle\langle m| \qquad (9)$$

that acts on the enlarged Hilbert space $\mathcal{H}\otimes\mathcal{H}_i$ including the signal Hilbert space $\mathcal{H}$ and the space $\mathcal{H}_i$ of an additional idler mode. An auxiliary idler mode is needed in order to preserve unitarity [16]: as we will see in the following, the idler mode is responsible for ''mixing'' the state, as in Eq. (3). The amplifying maps $\mathcal{A}_g$ and $\mathcal{A}_g^{\vee}$ pertain to the signal Hilbert space $\mathcal{H}$ only, and are obtained by partially tracing over the idler mode. One has

$$\mathcal{A}_g(|\psi\rangle\langle\psi|)=\mathrm{Tr}_i\{\hat{U}_g|\psi\rangle\langle\psi|\otimes\hat{\rho}_i\hat{U}_g^\dagger\}, \qquad (10)$$

$$\mathcal{A}_g^{\vee}[d\hat{\mu}(\phi)]=\mathrm{Tr}_i\{\hat{U}_g^\dagger\,d\hat{\mu}(\phi)\otimes\hat{1}\hat{U}_g\hat{1}\otimes\hat{\rho}_i\}, \qquad (11)$$

with $\hat{\rho}_i$ denoting the density matrix of the idler mode. From Eqs. (9) and (11) one can see that the ideal number deamplifier achieves the phase amplification given in Eq. (8) for any $\hat{\rho}_i$, with

$$\hat{\zeta}_g=g^{-1}\sum_{\lambda=0}^{g-1}\hat{\zeta}_g^{(\lambda)}, \qquad (12)$$

$$\hat{\zeta}_g^{(\lambda)}=\sum_{n,m=0}^{\infty}e^{2\pi ig^{-1}(n-m)\lambda}|n\rangle\langle[n/g]|\hat{\zeta}|[m/g]\rangle\langle m|, \qquad (13)$$

where $[x]$ denotes the integer part of x. The sum over λ accounts for the 2π periodicity. In fact, one has

$$\mathcal{A}_g^{\vee}[d\hat{\mu}(\phi)]=\frac{d\phi}{2\pi g}\sum_{\lambda=0}^{g-1}e^{ia^\dagger ag^{-1}(\phi+2\pi\lambda)}\hat{\zeta}_g^{(0)}$$
$$\times e^{-ia^\dagger ag^{-1}(\phi+2\pi\lambda)}, \qquad (14)$$

with $\hat{\zeta}_g^{(0)}$ given by Eq.(13). Notice that for the ideal POM (7) one has $\hat{\zeta}_g^{(0)}\equiv\hat{\zeta}$, and Eq. (14) is just a 2π-periodic rescaling of the POM: in this sense the present phase amplification can be considered ideal. It is also easy to check that the amplifier achieves the Schrödinger-picture transformation (3) independently on the idler state $\hat{\rho}_i$. One has

$$\hat{\rho}_{\psi}^{(g)} = \sum_{\nu=0}^{g-1} |\psi_{\nu}^{(g)}\rangle\langle\psi_{\nu}^{(g)}|, \quad |\psi_{\nu}^{(g)}\rangle = \sum_{n=0}^{\infty} \psi_{gn+\nu}|n\rangle, \quad (15)$$

where, for simplicity of notation, we retain unnormalized vectors $|\psi_{\nu}^{(g)}\rangle$.

IV. AMPLIFICATION OF PHASE-COHERENT STATES

From Eq.(15) it follows that $\hat{\rho}_{\psi}^{(g)}$ is pure only when $\psi_n \propto r^n$ for some constant r: hence, the only states which are not mixed by phase amplification—so providing optimum performance—are the so-called *phase-coherent* states [14]. These are defined as follows:

$$|\xi\rangle = (1-|\xi|^2)^{1/2} \sum_{n=0}^{\infty} \xi^n |n\rangle, \quad \xi = e^{i\varphi}|\xi|, |\xi|<1, \quad (16)$$

where the complex number ξ also carries the phase-shift information φ. Then, from Eqs.(10), (15), and (16) one has

$$\mathcal{A}_g^{\vee}(|\xi\rangle\langle\xi|) = |\xi^g\rangle\langle\xi^g|. \quad (17)$$

The phase-coherent state $|\xi\rangle$ has an average number of photons $\langle\hat{n}\rangle = |\xi|^2/(1-|\xi|^2)$. Notice that, apart from normalization, in the limit $|\xi|\to 1^-$ the state (16) approaches the Susskind-Glogower state

$$|e^{i\varphi}\rangle = \sum_{n=0}^{\infty} \exp(in\varphi)|n\rangle, \quad (18)$$

in terms of which the ideal POM can be rewritten as $d\hat{\mu}(\phi) = |e^{i\phi}\rangle\langle e^{i\phi}|\, d\phi/2\pi$: in this sense one can say that the phase-coherent states match the ideal POM for large number of photons. For ideal phase detection, the output phase probability after amplification is simply given by

$$p_{out}(\phi|\varphi) = \frac{1}{2\pi}|\langle\xi^g|e^{i\phi}\rangle|^2 = \frac{1}{2\pi}\frac{1-|\xi|^{2g}}{1+|\xi|^{2g}-2|\xi|^g\cos(\phi-g\varphi)}. \quad (19)$$

In the limit $|\xi|\to 1^-$ one has $p_{out}(\phi)\to\delta_{2\pi}(\phi-g\varphi)$, $\delta_{2\pi}$ denoting the periodicized δ. All quantities of interest can be evaluated analytically for $|\xi|=1-\epsilon$ with $g\epsilon \ll 1$ and $g\varphi \in [-\pi,\pi]$. The average phase is amplified as

$$\langle\phi\rangle_{out} = g\varphi + O(g\epsilon), \quad (20)$$

whereas the root-mean-square (rms) fluctuations

$$\langle\Delta\phi^2\rangle_{out} = 2g\epsilon + O(g^2\epsilon^2) \quad (21)$$

are amplified by only a factor g. Thus if we define the noise figure

$$\mathcal{R} = \frac{(\mathcal{S}^2/\mathcal{N})_{in}}{(\mathcal{S}^2/\mathcal{N})_{out}}, \quad (22)$$

where $\mathcal{S}$ and $\mathcal{N}$ denote, respectively, the signal $\langle\phi\rangle$ and the noise $\langle\Delta\phi^2\rangle$, we have

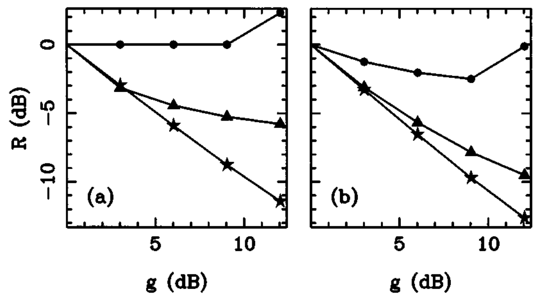

FIG. 1. Noise figure vs gain g (integer powers of 2) for input phase-coherent state (stars), coherent state (circles), and squeezed state (triangles) with $\langle\hat{n}\rangle_{in}=40$ and $\varphi=0.05$ for all states (the squeezed state has 2.6 squeezing photons: see [18]). (a): ideal detection. (b): heterodyne detection.

$$\mathcal{R} = \frac{1}{g}. \quad (23)$$

Actually, the same result would be obtained for almost all definitions of phase noise $\mathcal{N}$ that have been adopted in the literature in order to account for 2π periodicity (for a survey see Ref. [17]): one always has $\mathcal{N}\sim g$, apart from the choice $\mathcal{N} \doteq -\langle \ln p(\phi)\rangle \sim 1+\ln g/\ln(2\pi/\langle\hat{n}\rangle)$, which behaves even better. We take this result only as a preliminary indication of the goodness of the amplifier, and in Sec. V we will carefully study the efficiency of the amplifier on the basis of both BER and mutual information. On the other hand, the rough argument based on low noise figures can help us to understand easily how an ideal number deamplifier can work very efficiently for phase-coherent states. This is simply due to the fact that phase-coherent states exhibit shot noise $\langle\Delta\phi^2\rangle \propto \langle\hat{n}\rangle^{-1}$ and, at the same time, they are preserved under amplification. Hence, when deamplifying $\langle\hat{n}\rangle$ one gets a phase noise which is amplified by only a factor g. This is apparent in Fig. 1, where we compare the phase amplifier performance for input phase-coherent states with phase amplification of coherent and squeezed states.

V. PHASE MEASUREMENTS BASED ON BINARY HYPOTHESIS TESTING

Typically, the situation in which one takes advantage of amplification occurs when the signal is very low, below the detection threshold, and the amplifier is used to enhance the signal above the threshold. However, as amplification also increases noise, the net benefit must be evaluated carefully, by comparing the values of BER and mutual information [19] obtained with and without amplification. A paradigmatic situation is sketched in Fig. 2, where we consider a binary channel that pertains to the phase detection of a small signal, i.e., a small phase shift. The measurement consists of testing the hypothesis that a phase-shifting event has occurred, assigning the ''true'' value to every outcome above the threshold φ_s. The phase probability distributions corresponding to zero shift and to φ shift are depicted in different gray colors: they correspond to the reference zero-phase state $|''0''\rangle \equiv |\psi\rangle$ and to the shifted state $|''1''\rangle \equiv \exp(ia^\dagger a\varphi)|\psi\rangle$, respectively. The input signal is

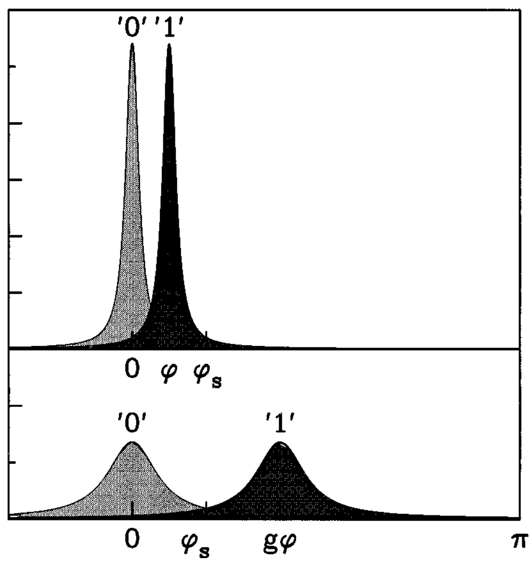

FIG. 2. Illustration of a binary hypothesis testing based on a phase measurement (see text). The top figure shows the phase probabilities corresponding to no event (''0'') and to the occurrence of a phase-shifting event (''1''). The bottom figure gives the same probabilities after phase amplification. Here, for the sake of pictorial representation we consider values that are not realistic for a high-sensitivity phase measurements ($\langle \hat{n} \rangle_{in}=8$, $\varphi=0.3$, $\varphi_s=0.6$, $g=4$: ideal detection of phase-coherent states).

very weak ($\varphi \ll 1$): the threshold φ_s is taken above φ due to limitations of the detector sensitivity and in order to achieve a low value of the ''false alarm probability'' $Q_{1|0}$ [6,19]

$$Q_{1|0}=\int_{\varphi_s}^{\pi} d\phi p(\phi|0), \tag{24}$$

namely, the probability of detecting ''1'' given state $|``0''\rangle$. It is clear that amplification will increase $Q_{1|0}$ as a consequence of the spread of the right tail of the ''0'' distribution; however, it will simultaneously enhance the ''detection probability'' $Q_{1|1}$ [6,19]

$$Q_{1|1}=\int_{\varphi_s}^{\pi} d\phi p(\phi|\varphi), \tag{25}$$

namely, the probability that ''1'' is correctly detected given state $|``1''\rangle$. An improvement of the binary test measurement is determined by a decrease of the bit-error rate

$$B=1+Q_{1|0}-Q_{1|1}, \tag{26}$$

or, equivalently, by an enhancement of the mutual information [19]

$$I=\sum_{j,k=0}^{1} p_j Q_{k|j} \ln\frac{Q_{k|j}}{\sum_{i=0}^{1} p_i Q_{k|i}}, \tag{27}$$

after specifying the *a priori* probabilities $\{p_j\}_{j=0,1}$ of input states $|``j''\rangle$, and considering all possible conditional probabilities $Q_{k|j}$ of detecting ''$k$'' given $|``j''\rangle$ (the probabilities $Q_{0|0}=1-Q_{1|0}$ and $Q_{0|1}=1-Q_{1|1}$ are just complementary of the previous ones).

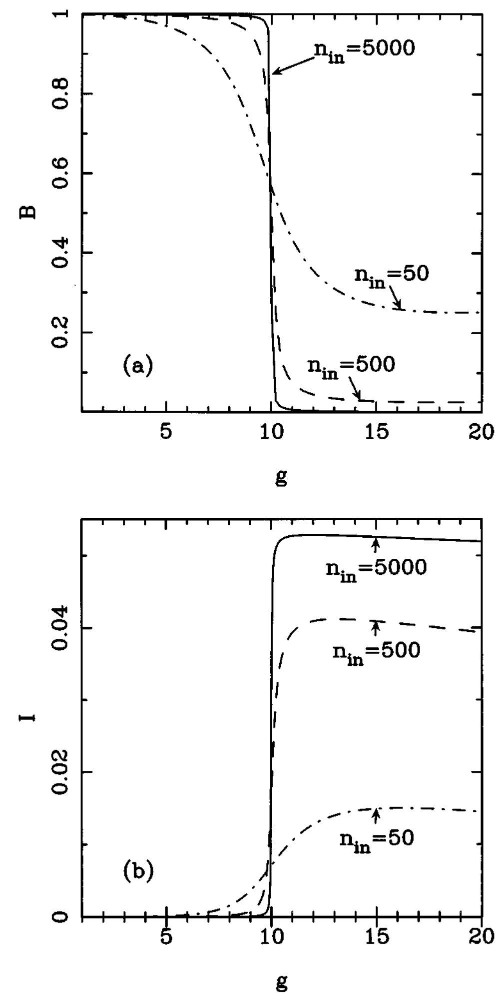

FIG. 3. Bit-error rate (a) and mutual information (b) vs gain g for phase-coherent states with input number of photons $\langle \hat{n} \rangle_{in}=50{,}500{,}5000$. The phase shift is $\varphi=0.05$, whereas the threshold phase is $\varphi_s=0.5$. For the mutual information a probability $p_1=0.01$ has been used (rare events).

A. Phase-coherent states

We now evaluate the conditional probabilities $Q_{1|0}$ and $Q_{1|1}$ with ($g>1$) and without ($g=1$) amplification. For $g\epsilon \ll 1$ we have

$$Q_{1|0}=\frac{g\epsilon}{2\pi}\cot\left(\frac{\varphi_s}{2}\right), \tag{28}$$

$$Q_{1|1}=\frac{1}{\pi}\left\{\frac{\pi}{2}-\arctan\left[\frac{2}{g\epsilon}\tan\left(\frac{\varphi_s-g\varphi}{2}\right)\right]\right\}. \tag{29}$$

These probabilities give the BER and the mutual information plotted in Fig. 3 as a function of the gain for different values of input number of photons $\langle \hat{n} \rangle_{in}$. The case of a very weak input signal $\varphi \ll \varphi_s$ has been considered. One can see that the BER exhibits a steep decrease and that, at the same time, the mutual information shows a rapid increase near the gain $g_s = \varphi_s/\varphi$. These features are further enhanced when the mean input photon number is increased. For the mutual information we refer to the situation of rare events, i.e., $p_1 = 1 - p_0 \ll 1$, which is of interest, for example, in interferometric detection of gravitational waves: the behavior of I, however, does not qualitatively depend on p_1, apart from the range of variation. On the other hand, if the input signal is above the detection threshold, i.e., $\varphi > \varphi_s$, one could see that the mutual information would monotonically decrease versus g, whereas there would be essentially no reduction of the BER. This is due to the fact that in this case amplification decreases the detection probability $Q_{1|1}$ given in Eq. (29).

B. Coherent states

Phase-coherent states are the only coherence preserving states under phase amplification. For generic input states phase amplification changes the kind of state and partially destroys coherence: for example, phase amplification does not preserve coherent or squeezed states. However, this does not mean that for such states the amplifier cannot improve the phase-shift measurement (on this subject, a preliminary indication is found in Fig. 1). Especially for nonideal phase detection, one can gain much benefit from phase amplification, also because the amplifier partially recovers the effective loss due to nonideal measurement. As an example, in Fig. 4 we have considered the realistic case of heterodyne phase detection of coherent states: here, the BER and the mutual information are plotted in the same fashion of Fig. 3 and for the same values of parameters φ, φ_s, and $\langle \hat{n} \rangle_{in}$. One can see that the amplifier works effectively, almost as well as for phase-coherent states. The only negative features are that the variations of B and I are less steep, and the amplifier efficiency is much reduced for low numbers of input photons. These phenomena are distinctive of a partial loss of coherence of the amplified state.

We emphasize that phase amplification is advantageous only for measurements of small phase shifts φ, and not too large gains g, such that $g\varphi \ll 1$. In fact, the transformation (14) folds the probability distribution at the boundaries of the 2π window in order to maintain the distribution as 2π periodic after the stretching along the direction of abscissa. In this way, in the limit of large gains any probability distribution would converge to the uniform probability on the 2π window.

In conclusion of this section some comments are in order regarding the apparent violation of the data processing theorem [19] regarding the improvement of mutual information. Indeed the theorem states the impossibility of improving the mutual information by performing any kind of data processing. More precisely, for a channel described by a map $X \to Y$ between input-output random variables X and Y, the mutual information $I(X|Y)$ between X and Y cannot be improved neither by any kind of "encoding" $U \to X$, nor by any "decoding" $Y \to V$, where U and V are additional random variables. In other words: $I(U|V) \leq I(X|Y)$, i.e., the end-to-end mutual information of the long Markov chain $U \to X \to Y \to V$ is never greater than that of the short chain $I(X|Y)$. The data processing theorem does not pertain to the present case of insertion of a quantum amplifier in a channel

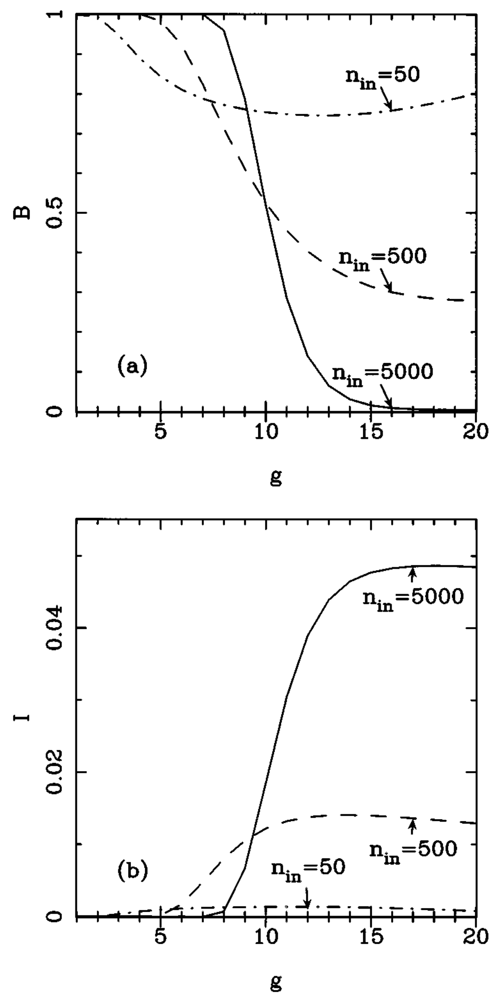

FIG. 4. Bit-error rate (a) and mutual information (b) vs gain g for coherent states and heterodyne detection (same values of parameters as in Fig. 3).

for the following two reasons. On the one hand, the amplifier is not used neither as an encoder, nor as a decoder—i.e., at one of the two ends of the chain—but is *inserted* in the chain as a *preamplifier* before a source of additive noise. If the amplifier admits a classical description in terms of an input-output probability map, then the insertion of the amplifier would correspond to changing the Markov chain $X \to Y$ to $X \to V \to Y$—namely, to changing the map $X \to Y$ instead of adding another data processing at one end of the chain: hence, the conditions for the data processing theorem do not apply. On the other hand, the amplifier is not a "classical" data processor, i.e., it is not equivalent to a measurement followed by data processing: coherence is only partially destroyed throughout the amplification process, and hence the amplifier is described by a map between probability amplitudes, rather than by a map between input-output probabilities. Probabilities are determined only at the very end of the chain, and depend on the observable that is measured at the output. In the quantum description, in addition to the input-output random variables X and Y we need to specify the detection POM $d\hat{\mu}(Y)$ at the end of the channel and the quantum state $\hat{\rho}_X$ encoding the input variable X, such that the probability map $X \to Y$ is given by the output conditional probability density $p(Y|X)dY = \mathrm{Tr}[\hat{\rho}_X d\hat{\mu}(Y)]$. Hence, the mutual information of the quantum channel can be denoted as $I[X,\hat{\rho}_X;Y,d\hat{\mu}(Y)]$. The insertion of a device in the quantum channel is described by a trace-preserving completely positive (CP) map $\hat{\rho}_X \to \mathcal{A}(\hat{\rho}_X)$ or by its dual $d\hat{\mu}(Y) \to \mathcal{A}^{\vee}[d\hat{\mu}(Y)]$. In general, if the system is suboptimal (i.e., the information is not optimized over the detection POM), there is no fixed inequality between $I[X,\hat{\rho}_X;Y,d\hat{\mu}(Y)]$ and $I[X,\mathcal{A}(\hat{\rho}_X);Y,d\hat{\mu}(Y)]$: the quantum device described by the CP map $\mathcal{A}$ reshapes the channel (i.e., leads to a different conditional probability between X and Y) with the possibility of improving the mutual information and approaching conditions for optimality. This situation corresponds to our case, where the system is suboptimal. If one wants to recover a situation closer to the one of the classical data processing theorem, one should optimize the mutual information over the detection POM, and the quantum analog of the data processing theorem can be written as follows:

$$\max_{d\hat{\mu}(Y)} I[X,\mathcal{A}(\hat{\rho}_X);Y,d\hat{\mu}(Y)] \leq \max_{d\hat{\mu}(Y)} I[X,\hat{\rho}_X;Y,d\hat{\mu}(Y)]. \quad (30)$$

The measurement-optimized system then cannot be further improved by the insertion of another device.

VI. EXPERIMENTAL REALIZATION OF THE SCHEME

Before concluding, some comments regarding the generation of phase-coherent states and the practical feasibility of photon number deamplifiers are in order.

Phase-coherent states can be ideally achieved using a PIA and a PND in series, as shown in Ref. [20]. In fact, the unitary evolution operator of the PIA is

$$\hat{U}_{\mathrm{PIA}} = \exp[\zeta a^\dagger b^\dagger - \overline{\zeta a} b], \quad (31)$$

where a and b describe signal and idler modes. When both modes are in the vacuum state at the input, the state at the output is

$$\hat{U}_{\mathrm{PIA}}(|0\rangle \otimes |0\rangle) = (1-|\xi|^2)^{1/2} \sum_{n=0}^{\infty} \xi^n |n\rangle \otimes |n\rangle, \quad (32)$$

where $\xi = \zeta/|\zeta| \tanh|\zeta|$. Then the twin beams are ideally converted into the one-mode phase-coherent state (16) using the PND in the inverse way. Concretely, the PND evolution can be well approximated by a sum-frequency up converter, described by the interaction Hamiltonian

$$\hat{H} = k(abc^\dagger + a^\dagger b^\dagger c). \quad (33)$$

For c initially in the vacuum state the performance of the sum-frequency converter very nearly approaches an ideal inverse PND [21]. The best approximation corresponds to maximum conversion for the mean photon number from modes a and b to mode c: the conversion time can be estimated to be $t_c \sim (1/2k)\langle\hat{n}\rangle_{in}^{-1/2} \ln\langle\hat{n}\rangle_{in}$ [22], where $\langle\hat{n}\rangle_{in}$ is the input mean photon number of either mode a or b. A sample of the phase probability distribution obtained from the frequency converter is given in Fig. 5, where it is compared with the perfect phase-coherent probability obtained with an ideal PND.

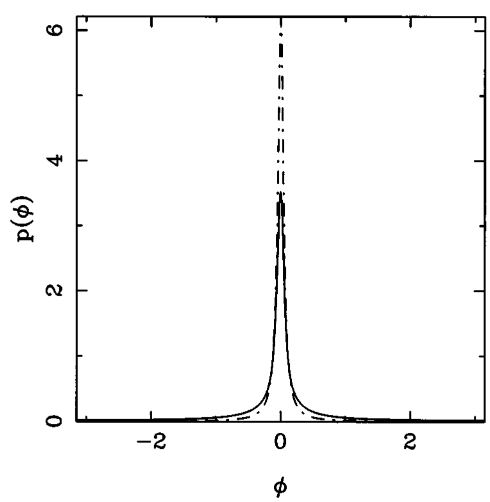

FIG. 5. Phase probability distribution of a phase-coherent state compared with the probability of a state achieved using a PIA and a sum frequency up converter in series (the sharper probability refers to the ideal state). The resulting average photon number is $\langle\hat{n}\rangle = 9.26$.

Regarding the feasibility of the photon number deamplifier (9), in Ref. [21] it has been shown that it is well approximated by g-order harmonic up conversion, and the situation is similar to the PND. More generally, Ref. [21] shows that both PNA and PND used in the inverse way (as in our case) are well approximated by up-conversion processes, whereas the direct operation is not well approached by the corresponding down-conversion processes, due to the incomplete depletion of the quantum pump.

VII. CONCLUSIONS

In conclusion, we have proposed a scheme for amplifying small phase shifts which reduces the BER and increases the information retrieved from the measurement. The best performance is achieved by phase-coherent states, but good results are also obtained in the practical situation of coherent states with heterodyne phase detection. We have shown how the PNA and PND—both originally proposed for matching direct detection—can be profitably used also for phase detection. When used as an ideal number deamplifier, the PNA becomes a phase amplifier that achieves ideal amplification independently on the state of the idler mode. The feasibility of both phase-coherent state generation and ideal number deamplification has been analyzed, based on phase-insensitive amplification, sum-frequency up conversion, and g-harmonic generation.

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[8] Notice the difference between: (i) non-negative operator and (ii) operator having only non-negative matrix elements in the number representation. Case (i) means that all diagonal matrix elements are non-negative in any representation. Hence, in general it is possible to have operators satisfying (ii) but not (i) and, conversely, operators satisfying (i) but not (ii).
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