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Probabilistic theories: What is special about Quantum Mechanics?

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To my friend and mentor, Professor Attilio Rigamonti.

Unperformed experiments have no results.
Asher Peres

5.1 Introduction

More than a century after its birth, quantum mechanics (QM) remains mysterious. We still don't have general principles from which to derive its remarkable mathematical framework, as happened for the amazing Lorentz transformations, which were rederived by Einstein from the invariance of physical laws in inertial frames and from the constancy of the speed of light.

Despite the utmost relevance of the problem of deriving QM from operational principles, research efforts in this direction have been sporadic. The deepest of the early attacks on the problem were the works of Birkhoff, von Neumann, Jordan, and Wigner, attempting to derive QM from a set of axioms with as much physical significance as possible [1, 2]. The general idea in Ref. [1] is to regard QM as a new kind of *prepositional calculus*, a proposal that spawned the research line of *quantum logic*, which is based on von Neumann's observation that the two-valued observables – represented in his formulation of QM by orthogonal projection operators – constitute a kind of "logic" of experimental propositions. After a hiatus of two decades of neglect, interest in quantum logic was revived by Varadarajan [3], and most notably by Mackey [4], who axiomatized QM within an operational framework, with the single exception of an admittedly *ad hoc* postulate, which represents the propositional calculus mathematically in the form of an orthomodular lattice. The most significant extension of Mackey's work is the general representation theorem of Piron [5].

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In the early work [2], Jordan, von Neumann, and Wigner considered the possibility of a commutative algebra of observables, with a product that needs only to define squares and sums of observables – the so-called *Jordan product* of observables a and b: $a \circ b := (a+b)^2 - a^2 - b^2$. However, such a product is generally non-associative and non-distributive with respect to the sum, and the quantum formalism follows only with additional axioms with no clear physical significance – e.g., a distributivity axiom for the Jordan product. Segal [6] later constructed an (almost) fully operational framework (with no experimental definition of the sum of observables) that allows generally non-distributive algebras of observables, but with a resulting mathematical framework largely more general than QM. As a result of this line of investigation, the purely algebraic formulation of QM gained in popularity versus the original Hilbert-space axiomatization.

In the algebraic axiomatization of QM, a physical system is defined by its C^*-algebra of observables (with identity), and the states of the system are identified with normalized positive linear functionals over the algebra, corresponding to the probability rules of measurements of observables. Indeed, the C^*-algebra of observables is more general than QM, since it includes classical mechanics as a special case, and generally describes any quantum–classical hybrid, thus being equivalent to QM with super-selection rules. Since in practice two observables are not distinguishable if they always exhibit the same probability distributions, at the operational level one can always take the set of states as *observable-separating* – in the sense that there are no different observables having the same probability distribution for all states. Conversely the set of observables is *state-separating*, i.e., there are no different states corresponding to the same probability distribution for all observables. Notice that, in principle, there exist different observables with the same expectation for all states, but higher moments will be different.[1]

The algebra of observables is generally considered to be more "operational" than the usual Hilbert-space axiomatization; however, little more is gained than a representation-independent mathematical framework. Indeed, the algebraic framework is unable to provide operational rules for how to measure sums and products of non-commuting observables.[2] The sum of two observables cannot be given an operational meaning, since a procedure involving the measurements of the two addenda would unavoidably assume that their respective measurements are jointly executable on the same system – i.e., the observables are *compatible*. The same

[1] This is not the case when one considers only *sharp observables*, for which there always exists a state such that the expectation of any function of the observable equals the function of the expectation. However, operationally we cannot rely on such a concept to define the general notion of an observable, since we cannot reasonably assume its feasibility (actual measurements are non-sharp).

[2] The spectrum of the sum is generally different from the sum of the spectra of the addenda, e.g., the spectra of xp_y and yp_x are both $\mathbb{R}$, whereas the angular-momentum component $xp_y - yp_x$ has a discrete spectrum. The same is true for the product.

reasoning holds for the product of two observables. A sum-observable defined as the one having expectation equal to the sum of expectations for all states [7] is clearly not unique, due to the existence of observables having the same expectation for all states, but with different higher moments. The only well-defined procedures are those involving single observables, such as the measurement of a *function of a single observable*, which operationally consists in just taking the function of the outcome.

The Jordan symmetric product has been regarded as a great advance in view of an operational axiomatization, since, in addition to being Hermitian (observables are Hermitian), it is defined only in terms of squares and sums of observables – i.e., without products. The definition of $a \circ b$, however, still relies on the notion of a sum of observables, which has no operational meaning. Remarkably, Alfsen and Shultz [8, 9] succeeded in deriving the Jordan product from solely geometrical properties of the convex set of states – e.g., orientability and faces shaped as Euclidean balls – however, again with no operational meaning. The problem with the Jordan product is that, in addition to not necessarily being associative, it is not even distributive, as the reader can easily check. It turns out that, modulo a few topological assumptions, the Jordan algebras can be embedded in the algebra $\mathsf{Lin}(\mathsf{H})$ of operators over the Hilbert space H, whereby $a \circ b = ab + ba$. Such assumptions, however, are still not operational. For a further critical overview of these earlier attempts at an operational axiomatization of QM, the reader is also directed to the recent books of Strocchi [7] and Thirring [10].

After a long suspension of research on the axiomatic approach – notably interrupted by the work of Ludwig and his school [11] – in the last few years the new field of quantum information has renewed interest in the problem of operational axiomatization of QM, having been boosted by the new experience on multipartite systems and *entanglement*. In his seminal paper [12] Hardy derived QM from five "reasonable axioms," which, more than being truly operational, are motivated on the basis of simplicity and continuity criteria, with the assumption of a finite number of perfectly discriminable states. His axiom 4, however, is still purely mathematical, and is directly related to the tensor-product rule for composite systems. In another popular paper [13], Clifton, Bub, and Halvorson have shown how three fundamental information-theoretic constraints – (a) the no-signaling constraint, (b) the no-broadcasting constraint, and (c) the impossibility of unconditionally secure bit commitment – suffice to entail that the observables and state space of a physical theory are quantum mechanical. Unfortunately, the authors started from a C^*-algebraic framework for observables, which, as already discussed, has little operational basis, and already coincides with the quantum–classical hybrid. Therefore, more than deriving QM, their informational principles just force the C^*-algebra of observables to be non-Abelian.

In Ref. [14][3] I showed how it is possible to derive the formulation of QM in terms of observables represented as Hermitian operators over Hilbert spaces with the right dimensions for the tensor product, starting from a few operational axioms. However, it is not clear yet whether such a framework is sufficient to identify QM (or the quantum–classical hybrid) as the only probabilistic theory resulting from axioms. Later, in Refs. [17–19], I showed how a C^*-algebraic framework for transformations (not for observables!) naturally follows from an operational probabilistic framework.

A very recent and promising direction for attacking the problem of QM axiomatization consists in positioning QM within the landscape of general probabilistic theories, including theories with non-local correlations stronger than the quantum ones, e.g., for the Popescu–Rohrlich boxes (PR boxes) [20]. Such theories have correlations that are "stronger" than the quantum ones – in the sense that they violate the quantum Cirel'son bound [21] – although they are still non-signaling, thus revealing the fortuitousness of the peaceful coexistence of QM and special relativity, in contrast with the claimed implication of QM linearity from the no-signaling condition [22]. Within the framework of the PR boxes general versions of the no-cloning and no-broadcasting theorems have been proved [23]. In Ref. [24] it has been shown that certain features generally thought of as specifically quantum are indeed present in all except classical theories. These include the non-unique decomposition of a mixed state into pure states, disturbance on measurement (related to the possibility of secure key distribution), and the no-cloning constraint. More recently, necessary and sufficient conditions have been established for teleportation [25], i.e., for reconstructing the state of a system on a remote identical system, using only local operations and joint states. In all these works quantum information has inspired the consideration of task-oriented axioms in a general operational framework that can incorporate QM, classical theory, and other non-signaling probabilistic theories (for an illustration of this general point of view see also Ref. [26]).

In this chapter I will consider the possibility of deriving QM as the mathematical representation of a *fair operational framework*, i.e., a set of rules that allows the experimenter to make predictions regarding future *events* on the basis of suitable *tests*, in a spirit close to Ludwig's axiomatization [11]. *States* are simply the compendia of probabilities for all possible outcomes of any test. I will consider a very general class of probabilistic theories, and examine the consequences of two postulates that need to be satisfied by any fair operational framework:

NSF: *no signaling from the future*, implying that it is possible to make predictions based on present tests;

[3] Most of the results of Ref. [14] were originally conjectured in Refs. [15] and [16].

PFAITH: *existence of preparationally faithful states*, implying the possibility of preparing any state and calibrating any test.

NSF is implicit in the very definition of conditional probabilities for cascade tests, entailing that *events are identified with transformations*, whence *evolution is identified with conditioning*. As we will see, such identifications lead to the notion of *effect* of Ludwig, i.e., the equivalence class of events occurring with the same probability for all states. I will show how we can introduce operationally a linear-space structure for effects. I will then show how all theories satisfying NSF admit a C^*-algebra representation of events as linear transformations of effects.

On the basis of a very general notion of dynamical independence, entailing the definition of a *marginal state*, it is immediately seen that all these theories are *non-signaling*, which is the current way of saying that the theories satisfy the principle of *Einstein locality*, namely that there can be no detectable effect on a system due to anything done to another non-interacting system. This is clearly another requirement for a fair operational framework. Postulate PFAITH then implies the *local observability principle*, namely the possibility of achieving an informationally complete test using only local tests – another requirement for a fair operational framework. The same postulate also implies many other features that are typically quantum, such as the tensor-product structure for the linear spaces of states and effects, the isomorphism of cones of states and effects (a weaker version of quantum self-duality), the so-called EPR cheating in bit commitment (which in Ref. [13], we remind the reader, was itself used as a postulate to derive QM), and many more. Dual to Postulate PFAITH an analogous postulate for effects would give additional quantum features, such as teleportation. However, all possible consequences of these postulates still need to be investigated, and it is not clear yet whether one can derive QM from these principles only.

In order to provide a route for seeking new candidates for operational postulates one can short-circuit the axiomatic framework to select QM using a mathematical postulate dictated by what is really special about the quantum theory, namely that not only transformations but also *effects form a C^*-algebra* (more precisely, this is true for all hybrid quantum–classical theories, i.e., those corresponding to QM plus super-selection rules). However, whereas the sum of effects can be operationally defined, their composition has no operational meaning, since the notion itself of "effect" abhors any kind of composition. I will then show that with another natural postulate,

AE: atomicity of evolution,

together with the mathematical postulate

CJ: Choi–Jamiolkowski isomorphism [27, 28], *it is possible to identify effects with "atomic" events*, i.e., elementary events that cannot be refined as the union of events. Via the composition of atomic events we can then define the composition of effects, thus selecting the quantum–classical hybrid among all possible general probabilistic theories (including the PR boxes, which indeed satisfy both NSF and PFAITH).

The CJ isomorphism looks natural in an operational context, and it is hoped that it will be converted soon into an operational postulate.

The present operational axiomatization will adhere to the following three general principles:

(1) **(Strongly Copenhagen)** Everything is defined operationally, including all mathematical objects. Operationally indistinguishable entities are identified.
(2) **(Mathematical closure)** Mathematical completion is taken for convenience.
(3) **(Operational closure)** Every operational option that is implicit in the formulation is incorporated in the axiomatic framework.

An example of the application of the strongly-Copenhagen principle is the notion of *system*, which here I will identify with a collection of tests – the tests that can be performed over the system. A typical case of operational identification is that of events occurring with the same probability and producing the same conditioning. Another case is the statement that the set of states is separating for effects and vice versa. Examples of mathematical closure are the norm closure, the algebraic closure, and the linear span. It is unquestionable that these are always idealizations of operational limiting cases, or they are introduced just to simplify the mathematical formulation (e.g., real numbers versus the "operational" rational numbers). Operational completeness, on the other hand, does not affect the corresponding mathematical representation, since every incorporated option is already implicit in the formulation. This is the case, for example, for convex closure, closure under coarse-graining, etc., which are already implicit in the probabilistic formulation.

5.2 C^*-Algebra representation of probabilistic theories

5.2.1 Tests and states

A probabilistic operational framework is a collection of **tests**[4] $\mathbb{A}, \mathbb{B}, \mathbb{C}, \ldots$ each being a complete collection $\mathbb{A} = \{\mathscr{A}_i\}$, $\mathbb{B} = \{\mathscr{B}_j\}$, $\mathbb{C} = \{\mathscr{C}_k\}$, ... of mutually

[4] The present notion of test corresponds to that of **experiment** of Ref. [14]. Quoted from that reference: "An experiment on an object system consists in making it interact with an apparatus, which will produce one of a set of possible events, each one occurring with some probability. The probabilistic setting is dictated by the need of experimenting with partial *a priori* knowledge about the system (and the apparatus). In the logic of performing experiments to predict results of forthcoming experiments in similar preparations, the information gathered in an experiment will concern whatever kind of information is needed to make predictions, and this, by definition, is the *state* of the object system at the beginning of the experiment. Such information is gained from the knowledge of which transformation occurred, which is the 'outcome' signaled by the apparatus."

exclusive **events** $\mathscr{A}_i, \mathscr{B}_j, \mathscr{C}_k, \ldots$ occurring probabilistically[5]; events that are mutually exclusive are often called **outcomes**. The same event can occur in different tests, with occurrence probability independent of the test. A **singleton test** – also called a **channel** – $\mathbb{D} = \{\mathscr{D}\}$ is **deterministic**: it represents a non-test, i.e., a free evolution. The **union** $\mathscr{A} \cup \mathscr{B}$ of two events corresponds to the event in which either $\mathscr{A}$ or $\mathscr{B}$ occurred, but it is unknown which one. A **refinement** of an event $\mathscr{A}$ is a set of events $\{\mathscr{A}_i\}$ occurring in some test such that $\mathscr{A} = \cup_i \mathscr{A}_i$. The experiment $\mathbb{A}$ itself can be regarded as the deterministic event corresponding to the complete union of its outcomes, and when regarded as an event it will be denoted by the different notation $\mathscr{D}_{\mathbb{A}}$. The opposite event of $\mathscr{A}$ in $\mathbb{A}$ will be denoted as $\overline{\mathscr{A}} := \complement_{\mathbb{A}} \mathscr{A}$.[6] The union of events transforms a test $\mathbb{A}$ into a new test $\mathbb{A}'$, which is a **coarse-graining** of $\mathbb{A}$, e.g., $\mathbb{A} = \{\mathscr{A}_1, \mathscr{A}_2, \mathscr{A}_3\}$ and $\mathbb{A}' = \{\mathscr{A}_1, \mathscr{A}_2 \cup \mathscr{A}_3\}$. Vice versa, we will call $\mathbb{A}$ a **refinement** of $\mathbb{A}'$.

The **state** ω describing the preparation of the system is the probability rule $\omega(\mathscr{A})$ for any event $\mathscr{A} \in \mathbb{A}$ occurring in any possible test $\mathbb{A}$.[7] For each test $\mathbb{A}$ we have the completeness $\sum_{\mathscr{A}_j \in \mathbb{A}} \omega(\mathscr{A}_j) = 1$. States themselves are considered as special tests: the **state-preparations**.

5.2.2 Cascading, conditioning, and transformations

The **cascade** $\mathbb{B} \circ \mathbb{A}$ of two tests $\mathbb{A} = \{\mathscr{A}_i\}$ and $\mathbb{B} = \{\mathscr{B}_j\}$ is the new test with events $\mathbb{B} \circ \mathbb{A} = \{\mathscr{B}_j \circ \mathscr{A}_i\}$, where $\mathscr{B} \circ \mathscr{A}$ denotes the **composite event** $\mathscr{A}$ "followed by" $\mathscr{B}$ satisfying the following

Postulate NSF (No signaling from the future). *The marginal probability $\sum_{\mathscr{B}_j \in \mathbb{B}} \omega(\mathscr{B}_j \circ \mathscr{A})$ of any event $\mathscr{A}$ is independent of test $\mathbb{B}$, and is equal to the probability with no test $\mathbb{B}$, namely*

$$\sum_{\mathscr{B}_j \in \mathbb{B}} \omega(\mathscr{B}_j \circ \mathscr{A}) =: f(\mathbb{B}, \mathscr{A}) \equiv \omega(\mathscr{A}), \quad \forall \mathbb{B}, \mathscr{A}, \omega. \tag{5.1}$$

[5] Also A. Rényi [29] calls our test "experiment." More precisely, he defines an experiment $\mathbb{A}$ as the pair $\mathbb{A} = (\mathfrak{X}, \mathcal{A})$ made of the *basic space* $\mathfrak{X}$ – the collection of outcomes – and of the σ-algebra of events $\mathcal{A}$. Here, to decrease the mathematical load of the framework, we conveniently identify the experiment with the basic space only, and consider a different σ-algebra (e.g., a coarse-graining) as a new test made of new mutually exclusive events. Indeed, since we are considering only discrete basic spaces, we can put basic space and σ-algebra in one-to-one correspondence, by taking $\mathcal{A} = 2^{\mathfrak{X}}$ – the power set of $\mathfrak{X}$ – and, vice versa, $\mathfrak{X}$ as the collection of the minimal intersections of elements of $\mathcal{A}$.

[6] By adding the intersection of events, one builds up the full *Boolean algebra of events* (see, e.g., Ref. [29]).

[7] By definition the state is the collection of the variables of a system knowledge of which is sufficient to make predictions. In the present context, it allows one to predict the results of tests, whence it is the probability rule for all events in any conceivable test.

NSF is part of the very definition of test-cascade; however, we treat it as a separate postulate, since it corresponds to the **choice of the arrow of time**.[8] The interpretation of the test-cascade $\mathbb{B} \circ \mathbb{A}$ is that "test $\mathbb{A}$ can influence test $\mathbb{B}$ but not vice versa."[9] Postulate NSF allows one to define the conditioned probability $p(\mathscr{B}|\mathscr{A}) = \omega(\mathscr{B} \circ \mathscr{A})/\omega(\mathscr{A})$ of event $\mathscr{B}$ occurring conditionally on the previous occurrence of event $\mathscr{A}$. It also guarantees that the probability of $\mathscr{B}$ remains independent of the test $\mathbb{B}$ when conditioned.

Conditioning sets a new probability rule corresponding to the notion of a **conditional state** $\omega_{\mathscr{A}}$, which gives the probability that an event occurs, knowing that event $\mathscr{A}$ has occurred with the system prepared in the state ω, namely $\omega_{\mathscr{A}} \doteq \omega(\cdot \circ \mathscr{A})/\omega(\mathscr{A})$.[10] We can now regard the event $\mathscr{A}$ as transforming with probability $\omega(\mathscr{A})$ the state ω to the (unnormalized) state[11] $\mathscr{A}\omega$ given by

$$\mathscr{A}\omega := \omega(\cdot \circ \mathscr{A}). \tag{5.2}$$

Therefore, the notion of cascade and postulate NSF entail the identification

event $\equiv$ transformation,

which in turn implies the equivalence[12]

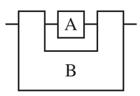
evolution $\equiv$ state-conditioning.

Notice that operationally a transformation $\mathscr{A}$ is completely specified by all the joint probabilities in which it is involved, whence it is unequivocally given by the probability rule $\mathscr{A}\omega = \omega(\cdot \circ \mathscr{A})$ for all states ω. This is equivalent to specifying both the conditional state $\omega_{\mathscr{A}}$ and the probability $\omega(\mathscr{A})$ for all possible states ω, due to the identity

$$\mathscr{A}\omega = \omega(\mathscr{A})\omega_{\mathscr{A}}. \tag{5.3}$$

[8] Postulate NSF is not just a Kolmogorov consistency condition for marginals of a joint probability. In fact, even though the marginal over test $\mathbb{B}$ in (5.1) is obviously the probability of $\mathscr{A}$, such probability in principle depends on the test $\mathbb{B}$, since the joint probability generally depends on it. Indeed, the marginal over entry $\mathscr{A}$ does generally depend on the past test $\mathbb{A} \ni \mathscr{A}$. Such asymmetry of the joint probability under marginalization over future or past tests represents *the choice of the arrow of time*. Of course one could have assumed the opposite postulate of no signaling from the past, considering conditioning from the future instead, thus reversing the arrow of time. Postulate NSF introduces conditioning from tests, and is part of the very definition of temporal cascade-tests. The need to consider NSF as a postulate was noticed for the first time by Masanao Ozawa (private communication).

[9] One could also define more general cascades not in time, e.g., the circuit diagram.



This would have given rise to a probabilistic version of the quantum comb theory of Ref. [30].

[10] Throughout, the central dot "$\cdot$" denotes the location of the pertinent variable.

[11] This is the same as the notion of *quantum operation* in QM, which gives the conditioning $\omega_{\mathscr{A}} = \mathscr{A}\omega/(\mathscr{A}\omega(\mathscr{I}))$, or, in other words, the analogue of the quantum Schrödinger-picture evolution of states.

[12] Clearly this includes the deterministic singleton-tests $\mathbb{D} = \{\mathscr{D}\}$ – the analogs of quantum channels, including unitary evolutions.

In particular the **identity transformation** $\mathscr{I}$ is completely specified by the rule $\mathscr{I}\omega = \omega$ for all states ω.

5.2.3 Systems

In a pure Copenhagen spirit we will identify a **system** S with a collection of **tests** $\mathsf{S} = \{\mathbb{A}, \mathbb{B}, \mathbb{C}, \ldots\}$, the collection being operationally closed under coarse-graining, convex combination, conditioning, and cascading, and will include all states as special tests. Closure under cascading is equivalent to considering **mono-systemic evolution**, i.e., in which there are only tests for which the output system is the same as the input one.[13] The operator has always the option of performing repeated tests, together with (randomly) alternating tests – say $\mathbb{A}$ and $\mathbb{B}$ – in different proportions – say p and $1-p$ $(0 < p < 1)$ – thus achieving the test $\mathbb{C}_p = p\mathbb{A} + (1-p)\mathbb{B}$ which is the **convex combination** of tests $\mathbb{A}$ and $\mathbb{B}$, and is given by $\mathbb{C}_p = \{p\mathscr{A}_1, p\mathscr{A}_2, \ldots, (1-p)\mathscr{B}_1, (1-p)\mathscr{B}_2, \ldots\}$, where $p\mathscr{A}$ is the same event as $\mathscr{A}$, but occurring with a probability rescaled by p. Since we will consider always closure under all the operator's options (this is our *operational closure*), we will take the system also to be closed under such convex combination. In particular, the set of all states of the system[14] is closed under convex combinations and under conditioning, and we will denote by $\mathfrak{S}(\mathsf{S})$ ($\mathfrak{S}$ for short) the convex set of all possible states of system S. We will often use the colloquialism "for all possible states ω" meaning $\forall\omega \in \mathfrak{S}(\mathsf{S})$, and we will do similarly for other operational objects.

In the following we will denote the set of all possible transformations/events by $\mathfrak{T}(\mathsf{S})$, $\mathfrak{T}$ for short. The convex structure of S entails a convex structure for $\mathfrak{T}$, whereas the cascade of tests entails the composition of transformations. The latter, together with the existence of the identity transformation $\mathscr{I}$, gives to $\mathfrak{T}$ the structure of a *convex monoid*.

5.2.4 Effects

From the notion of a conditional state two complementary types of equivalences for transformations follow: the *conditional* and the *probabilistic* equivalence. The transformations $\mathscr{A}_1$ and $\mathscr{A}_2$ are **conditioning-equivalent** when $\omega_{\mathscr{A}_1} = \omega_{\mathscr{A}_2} \forall\omega \in \mathfrak{S}$,

[13] We could have considered more generally tests in which the output system is different from the input one, in which case the system is no longer closed under a test-cascade, and, instead, there are cascades of tests from different systems. This would give more flexibility to the axiomatic approach, and could be useful for proving some theorems related to multipartite systems made of different systems. The fact that there are different systems would impose constraints on the cascades of tests, corresponding to allowing only some particular words made of the "alphabet" $\mathbb{A}, \mathbb{B}, \ldots$ of tests, and the system would then correspond to a "language" (see Ref. [31] for a similar framework). Such generalization will be thoroughly analyzed in a forthcoming publication.

[14] At this stage such a set does not necessarily contain all *in-principle* possible states. The extension will be done later, after defining effects.

namely when they produce the same conditional state for all prior states ω. On the other hand, the transformations $\mathscr{A}_1$ and $\mathscr{A}_2$ are **probabilistically equivalent** when $\omega(\mathscr{A}_1) = \omega(\mathscr{A}_2)\ \forall\omega \in \mathfrak{S}$, namely when they occur with the same probability.[15] Since operationally a transformation $\mathscr{A}$ is completely specified by the probability rule $\mathscr{A}\omega$ for all states, it follows that two transformations $\mathscr{A}_1$ and $\mathscr{A}_2$ are fully **equivalent** (i.e., operationally indistinguishable) when $\mathscr{A}_1\omega = \mathscr{A}_2\omega$ for all states ω. We will identify two equivalent transformations, and denote the equivalence simply as $\mathscr{A}_1 = \mathscr{A}_2$. From identity (5.3) it follows that two transformations are equivalent if and only if they are both conditioning and probabilistically equivalent.

A probabilistic equivalence class of transformations defines an **effect**.[16] In the following we will denote effects with lower-case letters $a, b, c, \ldots$ and denote by $[\mathscr{A}]_{\text{eff}}$ the effect containing transformation $\mathscr{A}$. We will also write $\mathscr{A} \in a$ meaning that "the transformation $\mathscr{A}$ belongs to the equivalence class a," or "$\mathscr{A}$ has effect a," and write "$\mathscr{A} \in [\mathscr{B}]_{\text{eff}}$" to say that "$\mathscr{A}$ is probabilistically equivalent to $\mathscr{B}$." Since by definition $\omega(\mathscr{A}) = \omega([\mathscr{A}]_{\text{eff}})$, hereafter we will legitimately write the variable of the state as an effect, e.g., $\omega(a)$. The **deterministic effect** will be denoted by e, corresponding to $\omega(e) = 1$ for all states ω. We will denote the set of effects for a system S as $\mathfrak{E}(\mathsf{S})$, or just $\mathfrak{E}$ for short. The set of effects inherits a convex structure from that of transformations.

By the same definition of state – as probability rule for transformations – states are separated by effects (whence also by transformations[17]), and, conversely, effects are separated by states. Transformations are separated by states in the sense that $\mathscr{A} \neq \mathscr{B}$ iff $\mathscr{A}\omega \neq \mathscr{B}\omega$ for some state. As a consequence, it may happen that the introduction of a new state via some new preparation (such as introducing additional systems) will separate two previously indiscriminable transformations, in which case we will include the new state (and all convex combinations with it) in $\mathfrak{S}(\mathsf{S})$, and we will complete the system S accordingly. We will end with $\mathfrak{S}(\mathsf{S})$ separating $\mathfrak{T}(\mathsf{S})$ and $\mathfrak{E}(\mathsf{S})$, and $\mathfrak{E}(\mathsf{S})$ separating $\mathfrak{S}(\mathsf{S})$.

The identity $\omega_{\mathscr{A}}(\mathscr{B}) \equiv \omega_{\mathscr{A}}([\mathscr{B}]_{\text{eff}})$ implies that $\omega(\mathscr{B}\circ\mathscr{A}) = \omega([\mathscr{B}]_{\text{eff}}\circ\mathscr{A})$ for all states ω, leading to the chaining rule $[\mathscr{B}]_{\text{eff}} \circ \mathscr{A} = [\mathscr{B} \circ \mathscr{A}]_{\text{eff}}$, corresponding to the "Heisenberg-picture" evolution in terms of transformations acting on effects.

15 In the papers [14–17] I called the conditional equivalence *dynamical equivalence*, since the two transformations will effect the same state change. However, one should more properly regard the "dynamical" change of the state ω due to the transformation $\mathscr{A}$ as the unnormalized state $\mathscr{A}\omega$, but the two transformations $\mathscr{A}$ and $\mathscr{B}$ will be fully equivalent when $\mathscr{A}\omega = \mathscr{B}\omega$ for all states ω. Moreover, in the same papers I called the probabilistic equivalence *informational equivalence*, since the two transformations will give the same information about the state. The new nomenclature has a more immediate meaning.

16 This is the same notion of "effect" introduced by Ludwig [11].

17 In fact, $\mathscr{A}\omega \neq \mathscr{A}\zeta$ for $\mathscr{A} \in \mathfrak{T}$ means that there exists an effect c such that $\mathscr{A}\omega(c) \neq \mathscr{A}\zeta(c)$, whence the effect $c \circ \mathscr{A}$ will separate the same states.

Notice that transformations act on effects from the right, inheriting the composition rule of transformations ($\mathscr{B} \circ \mathscr{A}$ means "$\mathscr{A}$ followed by $\mathscr{B}$"). Notice also that $e \circ \mathscr{A} \in [\mathscr{I} \circ \mathscr{A}]_{\text{eff}} = a$. It follows that for $\mathscr{D}$ deterministic one has $\mathscr{D} \in e$, whence $\mathscr{D} \circ \mathscr{A} \in [\mathscr{A}]_{\text{eff}}$.

Consistently, in the "Schrödinger picture," we have $\mathscr{B}\omega(\cdot \circ \mathscr{A}) = \omega(\cdot \circ \mathscr{B} \circ \mathscr{A})$, corresponding to $(\mathscr{B} \circ \mathscr{A})\omega = \omega(\cdot \circ \mathscr{B} \circ \mathscr{A})$. Also, we will use the unambiguous notation $\mathscr{B}\omega(a) = [\mathscr{B}\omega](a)$, whence $\mathscr{B}\omega(a) = \omega(a \circ \mathscr{B})$, and $\omega(a) = \mathscr{A}\omega(e)$, $\forall \mathscr{A} \in a$.

5.2.5 Linear structures for transformations and effects

Transformations $\mathscr{A}_1$ and $\mathscr{A}_2$, for which one has the bound $\omega(\mathscr{A}_1) + \omega(\mathscr{A}_2) \leqslant 1$, $\forall \omega \in \mathfrak{S}$, can in principle occur in the same test, and we will call them **test-compatible**. For test-compatible transformations one can define their addition $\mathscr{A}_1 + \mathscr{A}_2$ via the probability rule

$$(\mathscr{A}_1 + \mathscr{A}_2)\omega = \mathscr{A}_1\omega + \mathscr{A}_2\omega, \tag{5.4}$$

where we remind the reader that $\mathscr{A}\omega := \omega(\cdot \circ \mathscr{A})$. Therefore the sum of two test-compatible transformations is just the union-event $\mathscr{A}_1 + \mathscr{A}_2 = \mathscr{A}_1 \cup \mathscr{A}_2$, with the two transformations regarded as belonging to the same test.[18] For any test $\mathbb{A}$ we can define its **total coarse-graining** as the deterministic transformation $\mathscr{D}_{\mathbb{A}} = \sum_{\mathscr{A}_i \in \mathbb{A}} \mathscr{A}_i$. We can trivially extend the addition rule (5.4) to any set of (generally non-test-compatible) transformations, and to subtraction of transformations as well. Notice that the composition "$\circ$" is distributive with respect to addition "+."

We can define the multiplication $\lambda\mathscr{A}$ of a transformation $\mathscr{A}$ by a scalar $0 \leqslant \lambda \leqslant 1$ by the rule

$$\omega(\cdot \circ \lambda\mathscr{A}) = \lambda\omega(\cdot \circ \mathscr{A}), \tag{5.5}$$

namely $\lambda\mathscr{A}$ is the transformation conditioning-equivalent to $\mathscr{A}$, but occurring with rescaled probability $\omega(\lambda\mathscr{A}) = \lambda\omega(\mathscr{A})$ – as happens in the convex combination of tests. It follows that for every couple of transformations $\mathscr{A}$ and $\mathscr{B}$ the transformations $\lambda\mathscr{A}$ and $(1 - \lambda)\mathscr{B}$ are test-compatible for $0 \leqslant \lambda \leqslant 1$, consistently with the convex closure of the system S. By extending the definition (5.5) to any positive λ, we then introduce the cone $\mathfrak{T}_+$ of transformations. We will call an event $\mathscr{A}$ **atomic**

[18] The probabilistic class of $\mathscr{A}_1 + \mathscr{A}_2$ is given by

$$\omega(\mathscr{A}_1 + \mathscr{A}_2) = \omega(\mathscr{A}_1) + \omega(\mathscr{A}_2), \quad \forall \omega \in \mathfrak{S},$$

whereas the conditional class is given by

$$\omega_{\mathscr{A}_1+\mathscr{A}_2} = \frac{\omega(\mathscr{A}_1)}{\omega(\mathscr{A}_1 + \mathscr{A}_2)}\omega_{\mathscr{A}_1} + \frac{\omega(\mathscr{A}_2)}{\omega(\mathscr{A}_1 + \mathscr{A}_2)}\omega_{\mathscr{A}_2}, \quad \forall \omega \in \mathfrak{S}.$$

if it has no non-trivial refinement in any test, namely if it cannot be written as $\mathscr{A} = \sum_i \mathscr{A}_i$ with $\mathscr{A}_i \neq \lambda_i \mathscr{A}$ for some i and $0 < \lambda_i < 1$. Notice that the identity transformation is not necessarily atomic.[19] The set of extremal rays of the cone $\mathfrak{T}_+$ – denoted by $\mathsf{Erays}(\mathfrak{T}_+)$ – contains the atomic transformations.

The notions of (i) test-compatibility, (ii) sum, and (iii) multiplication by a scalar are naturally inherited from transformations to effects via probabilistic equivalence, and then to states via duality. Correspondingly, we introduce the cone of effects $\mathfrak{E}_+$, and, by duality, we extend the cone of states $\mathfrak{S}_+$ to the dual cone of $\mathfrak{E}_+$, completing the set of states $\mathfrak{S}$ to the cone-base of $\mathfrak{S}_+$ made of all positive linear functionals over $\mathfrak{E}_+$ normalized at the deterministic effect, namely all in-principle legitimate states (in parallel we complete the system S with the corresponding state-preparations). We call such a completion of the set of states the **no-restriction hypothesis for states**, corresponding to the **state–effect duality**, namely the convex cones of states $\mathfrak{S}_+$ and of effects $\mathfrak{E}_+$ are dual each other.[20] The state cone $\mathfrak{S}_+$ introduces a natural **partial ordering** $\geqslant$ over states and over effect (via duality), and one has $a \in \mathfrak{E}$ iff $0 \leqslant a \leqslant e$. Thus the convex set $\mathfrak{E}$ **is a truncation of the cone** $\mathfrak{E}_+$, whereas $\mathfrak{S}$ **is a base for the cone** $\mathfrak{S}_+$[21] defined by the normalization condition $\omega \in \mathfrak{S}$ iff $\omega \in \mathfrak{S}_+$ and $\omega(e) = 1$. In the following it will be useful also to express the probability rule $\omega(a)$ also in its dual form $a(\omega) = \omega(a)$, with the effect acting on the state as a linear functional.

By extending (5.5) to any real (complex) scalar λ we build the linear real (complex) span $\mathfrak{T}_\mathbb{R} = \mathsf{Span}_\mathbb{R}(\mathfrak{T})$ ($\mathfrak{T}_\mathbb{C} = \mathsf{Span}_\mathbb{C}(\mathfrak{T})$). The *Cartesian decomposition* $\mathfrak{T}_\mathbb{C} = \mathfrak{T}_\mathbb{R} \oplus \mathrm{i}\mathfrak{T}_\mathbb{R}$ holds, i.e., each element $\mathscr{A} \in \mathfrak{T}_\mathbb{C}$ can be uniquely written as $\mathscr{A} = \mathscr{A}_R + \mathrm{i}\mathscr{A}_I$, with $\mathscr{A}_R, \mathscr{A}_I \in \mathfrak{T}_\mathbb{R}$.[22] Analogously, also for effects and states we define $\mathfrak{E}_\mathbb{F}$, $\mathfrak{S}_\mathbb{F}$ for $\mathbb{F} = \mathbb{R}, \mathbb{C}$. The state–effect duality implies the linear space identifications $\mathfrak{S}_\mathbb{F} \equiv \mathfrak{E}_\mathbb{F}$. Thanks to such identifications and to the identity of the dimension of a convex cone with that of its complex and real spans, in the following, without ambiguity, we will simply write $\dim(\mathsf{S}) := \dim[\mathfrak{S}_+(\mathsf{S})] \equiv \dim[\mathfrak{E}_+(\mathsf{S})]$. Moreover, if there is no confusion, then with some abuse of terminology we will simply

[19] For example, the identity transformation is refinable in classical Abelian probabilistic theory, where states are of the form $\varrho = \sum_l p_l |l\rangle\langle l|$, with $\{|l\rangle\}$ a complete orthonormal basis and $\{p_l\}$ a probability distribution. Here the identity transformation is given by $\mathscr{I} = \sum_k |k\rangle\langle k| \cdot |k\rangle\langle k|$, $\{|k\rangle\}$, which is refinable into rank-one projection maps.

[20] In infinite dimensions one also takes the closure of cones.

[21] We remind the reader that a set $\mathsf{B} \subset \mathsf{C}$ of a cone C in a vector space V is called the *base* of C if $0 \notin \mathsf{B}$ and for every point $u \in \mathsf{C}$, $u \neq 0$, there is a unique representation $u = \lambda v$, with $v \in \mathsf{B}$ and $\lambda > 0$. Then, one has that $u \in \mathsf{C}$ spans an extreme ray of C iff $u = \lambda v$, where $\lambda > 0$ and v is an extreme point of B (see Ref. [32]).

[22] Note that the elements $\mathscr{T} \in \mathfrak{T}_\mathbb{R}$ can in turn be decomposed *à la* Jordan as $\mathscr{T} = \mathscr{T}_+ - \mathscr{T}_-$, with $\mathscr{T}_\pm \in \mathfrak{T}_+$. However, such a decomposition is generally not unique. According to a theorem of Béllissard and Jochum [33] the Jordan decomposition of the elements of the real span of a cone (with $\mathscr{T}_\pm$ orthogonal in $\mathfrak{T}_\mathbb{R}$ Euclidean space) is unique if and only if the cone is self-dual.

refer by "states," "effects," and "transformations" to the respective generalized versions that are elements of the cones, or of their real and complex linear spans.

Note that the cones of states and effects contain the origin, i.e., the null vector of the linear space. For the cone of states one has that $\omega = 0$ iff $\omega(e) = 0$ (since for any effect a one has $0 \leqslant \omega(a) \leqslant \omega(e) = 0$, namely $\omega(a) = 0$). On the other hand, the hyperplane which truncates the cone of effects giving the physical convex set $\mathfrak{E}$ is conveniently characterized using any **internal state** ϑ – i.e., a state that can be written as the convex combination of any state with some other state – by using the following lemma.

Lemma 1. *For any $a \in \mathfrak{E}_+$ one has $a = 0$ iff $\vartheta(a) = 0$ and $a = e$ iff $\vartheta(a) = 1$, with ϑ any internal state.*

Proof. For any state ω one can write $\vartheta = p\omega + (1 - p)\omega'$ with $0 \leqslant p \leqslant 1$ and $\omega' \in \mathfrak{S}$. Then one has $\vartheta(a) = 0$ iff $\omega(a) = 0$ $\forall\omega \in \mathfrak{S}$, that is iff $a = 0$. Moreover, one has $\vartheta(a) = 1$ iff $\omega(a) = 1$ $\forall\omega \in \mathfrak{S}$, i.e., $a = e$.

5.2.6 Observables and informational completeness

An **observable** $\mathbb{L}$ is a complete set of effects $\mathbb{L} = \{l_i\}$ summing to the deterministic effect as $\sum_{l_i \in \mathbb{L}} l_i = e$, namely l_i are the effects of the events of a test. An observable $\mathbb{L} = \{l_i\}$ is named **informationally complete** for S when each effect can be written as a real linear combination of l_i, namely $\mathsf{Span}_{\mathbb{R}}(\mathbb{L}) = \mathfrak{E}_{\mathbb{R}}(\mathsf{S})$. When the effects of $\mathbb{L}$ are linearly independent the informationally complete observable is named *minimal*. Clearly, since $\mathfrak{E}$ is separating for states, **any informationally complete observable separates states**, that is using an informationally complete observable we can reconstruct also any state $\omega \in \mathfrak{S}(\mathsf{S})$ from the set of probabilities $\omega(l_i)$. The existence of a minimal informationally complete observable constructed from the set of available tests is guaranteed by the following theorem.

Theorem 1. (Existence of minimal informationally complete observable). *It is always possible to construct a minimal informationally complete observable for* S *out of a set of tests of* S.

For the proof see Ref. [17].

In the following we will take a fixed minimal informationally complete observable $\mathbb{L} = \{l_i\}$ as a **reference test**, with respect to which all basis-dependent representations will be defined.

Symmetrically to the notion of an informationally complete observable we have the notion of a **separating set of states** $\mathbb{S} = \{\omega_i\}$, in terms of which one can write any state as a real linear combination of the states $\{\omega_i\}$, namely

$\mathfrak{S}_{\mathbb{R}}(\mathsf{S}) = \mathsf{Span}_{\mathbb{R}}(\mathbb{S})$. Regarded as a test $\mathbb{S} = \{\mathscr{S}_i\} \in \mathsf{S}$ the set of states $\{\omega_i\}$ corresponds to the state-reduction $\mathscr{S}_i\omega = \omega(\mathscr{S}_i)\omega_i$, $\forall\omega \in \mathfrak{S}$. When the corresponding effects $[\mathscr{S}_i]_{\text{eff}}$ form an informationally complete observable the test $\mathbb{S}$ would be an example of the *Quantum Bureau International des Poids et Mesures* of Fuchs [34].

5.2.7 Banach structures

On states $\omega \in \mathfrak{S}$ introduce the **natural norm** $\|\omega\| = \sup_{a\in\mathfrak{E}} \omega(a)$, which extends to the whole linear space $\mathfrak{S}_{\mathbb{R}}$ as $\|\omega\| = \sup_{a\in\mathfrak{E}} |\omega(a)|$. Then, we can introduce the dual norm on effects $\|a\| := \sup_{\omega\in\mathfrak{S}_{\mathbb{R}},\, \|\omega\|\leqslant 1} |\omega(a)|$, and then on transformations $\|\mathscr{A}\| := \sup_{b\in\mathfrak{E}_{\mathbb{R}}, \|b\|\leqslant 1} \|b\circ\mathscr{A}\|$. Closures in norm (for mathematical convenience) make $\mathfrak{E}_{\mathbb{R}}$ and $\mathfrak{S}_{\mathbb{R}}$ a dual Banach pair, and $\mathfrak{T}_{\mathbb{R}}$ a real Banach algebra.[23] Therefore, all operational quantities can be mathematically represented as elements of such Banach spaces.

5.2.8 The Metric

One can define a **natural distance** between states $\omega, \zeta \in \mathfrak{S}$ as follows:

$$d(\omega,\zeta) := \sup_{l\in\mathfrak{E}} l(\omega) - l(\zeta). \tag{5.6}$$

Lemma 2. *The function (5.6) is a metric on $\mathfrak{S}$, and is bounded as* $0 \leqslant d(\omega,\zeta) \leqslant 1$.

Proof. For every effect l, $e-l$ is also a effect, whence

$$\begin{aligned} d(\omega,\zeta) &= \sup_{l\in\mathfrak{E}}(l(\omega)-l(\zeta)) = \sup_{l'\in\mathfrak{E}}((e-l')(\omega)-(e-l')(\zeta)) \\ &= \sup_{l'\in\mathfrak{E}}(l'(\zeta)-l'(\omega)) = d(\zeta,\omega), \end{aligned} \tag{5.7}$$

that is, d is symmetric. On the other hand, $d(\omega,\zeta) = 0$ implies that $\zeta = \omega$, since the two states must give the same probabilities for all transformations. Finally, one has

$$\begin{aligned} d(\omega,\zeta) &= \sup_{l\in\mathfrak{E}}(l(\omega)-l(\theta)+l(\theta)-l(\zeta)) \\ &\le \sup_{l\in\mathfrak{E}}(l(\omega)-l(\theta)) + \sup_{l\in\mathfrak{E}}(l(\theta)-l(\zeta)) = d(\omega,\theta)+d(\theta,\zeta), \end{aligned} \tag{5.8}$$

[23] An algebra of maps over a Banach space inherits the norm induced by that of the Banach space on which it acts. It is then easy to prove that the closure of the algebra under such a norm is a Banach algebra.

that is, it satisfies the triangular inequality, whence d is a metric. By construction, the distance is bounded as $d(\omega, \zeta) \leqslant 1$, since the maximum value of $d(\omega, \zeta)$ is achieved when $l(\omega) = 1$ and $l(\zeta) = 0$.

The natural distance (5.7) is extended to a metric over $\mathfrak{S}_{\mathbb{R}}$ as $d(\omega, \zeta) = \|\omega - \zeta\|$ with $\|\cdot\|$ the norm over $\mathfrak{S}_{\mathbb{R}}$. Analogously we define the distance between effects as $d(a, b) := \sup_{\omega\in\mathfrak{S}} |\omega(a - b)|$.[24]

A relevant property of the metric in (5.6) is its **monotonicity**, namely that the distance between two states can never increase under deterministic evolution, as established by the following lemma.

Lemma 3. (Monotonicity of the state distance). *For every deterministic physical transformation $\mathscr{D} \in \mathfrak{T}$, one has*

$$d(\mathscr{D}\omega, \mathscr{D}\zeta) \leqslant d(\omega, \zeta). \tag{5.9}$$

Proof. First we notice that since $\mathscr{D} \in \mathfrak{T}$ is a physical transformation, for every effect $a \in \mathfrak{E}$ one has also $a \circ \mathscr{D} \in \mathfrak{E}$, whence $\mathfrak{E} \circ \mathscr{D} \subseteq \mathfrak{E}$. Therefore, we have

$$\begin{aligned} d(\mathscr{D}\omega, \mathscr{D}\zeta) :&= \sup_{a\in\mathfrak{E}} \omega(a \circ \mathscr{D}) - \zeta(a \circ \mathscr{D}) \\ &= \sup_{a\in\mathfrak{E}\circ\mathscr{D}} \omega(a) - \zeta(a) \leqslant \sup_{a\in\mathfrak{E}} \omega(a) - \zeta(a) = d(\omega, \zeta). \end{aligned} \tag{5.10}$$

Notice that we take the transformation deterministic only to assure that $\mathscr{D}\omega$ is itself a state for any ω.

5.2.9 Isometric transformations

A deterministic transformation $\mathscr{U}$ is called *isometric* if it preserves the distance between states, namely

$$d(\mathscr{U}\omega, \mathscr{U}\zeta) \equiv d(\omega, \zeta), \qquad \forall\omega, \zeta \in \mathfrak{S}. \tag{5.11}$$

Lemma 4. *In finite dimensions, all the following properties of a transformation are equivalent: (a) it is isometric for $\mathfrak{S}$; (b) it is isometric for $\mathfrak{E}$; (c) it is an automorphism of $\mathfrak{S}$; and (d) it is an automorphism of $\mathfrak{E}$.*

Proof. By definition a transformation of the convex set (of states or effects) is a linear map of the convex set in itself. A linear isometric map of a set in itself is isometric on the linear span of the set.[25] (Recall that the natural distance between

[24] It is easy to check that such a distance satisfies the trangular inequality.
[25] Interestingly, the Mazur–Ulam theorem states that any surjective isometry (not necessarily linear) between real-normed spaces is affine. Therefore, even if non-linear, it would map convex subsets to convex subsets.

states has been extended to a metric over the whole $\mathfrak{S}_{\mathbb{R}}$.) In finite dimensions an isometry on a normed linear space is diagonalizable [35]. Its eigenvalues must have unit modulus, otherwise it would not be isometric. It follows that it is an orthogonal transformation, and, since it maps the set into itself, it must be a linear automorphism of the set. Therefore, an isometric transformation of a convex set is an automorphism of the convex set.[26]

Now, automorphisms of $\mathfrak{S}$ are isometric for $\mathfrak{E}$, since

$$\begin{aligned} d(a \circ \mathscr{U}, b \circ \mathscr{U}) &= \sup_{\omega \in \mathfrak{S}} |\omega((a-b) \circ \mathscr{U})| = \sup_{\omega \in \mathfrak{S}} |(\mathscr{U}\omega)(a-b)| \\ &= \sup_{\omega \in \mathscr{U}\mathfrak{S}} |\omega(a-b)| = \sup_{\omega \in \mathfrak{S}} |\omega(a-b)| = d(a,b), \end{aligned} \tag{5.12}$$

and, similarly, automorphisms of $\mathfrak{E}$ are isometric for $\mathfrak{S}$, since

$$\sup_{a \in \mathfrak{E}} [\omega(a \circ \mathscr{U}) - \zeta(a \circ \mathscr{U})] = \sup_{a \in \mathfrak{E} \circ \mathscr{U}} [\omega(a) - \zeta(a)] = d(\omega, \zeta). \tag{5.13}$$

Therefore, automorphisms of $\mathfrak{S}$ are isometric for $\mathfrak{E}$, whence, for the first part of the proof, they are automorphisms of $\mathfrak{E}$, whence they are isometric for $\mathfrak{S}$.

The *physical automorphisms* play the role of unitary transformations in QM.

Corollary 1. (Wigner theorem). *The only transformations of states that are inverted by another transformation must send pure states to pure states, and are isometric.*

5.2.10 The C^*-algebra of transformations

We can represent the transformations as elements of $\mathfrak{T}_{\mathbb{C}}$ regarded as a complex C^*-algebra. This is obvious, since $\mathfrak{T}_{\mathbb{C}}$ are by definition linear transformations of effects, making an associative sub-algebra $\mathfrak{T}_{\mathbb{C}} \subseteq \mathsf{Lin}(\mathfrak{E}_{\mathbb{C}})$ of the matrix algebra over $\mathfrak{E}_{\mathbb{C}}$. The **adjoint** and **norm** can be easily defined in terms of any chosen **scalar product** $(\cdot,\cdot)$ **over** $\mathfrak{E}_{\mathbb{C}}$, with the adjoint defined as $(a \circ \mathscr{A}^\dagger, b) = (a, b \circ \mathscr{A})$, and the norm as $\|\mathscr{A}\| = \sup_{a \in \mathfrak{E}_{\mathbb{C}}} \|a \circ \mathscr{A}\|/\|a\|$, with $\|a\| = \sqrt{(a,a)}$. (Notice that these norms are different from the "natural norms" defined in Section 5.2.7.) We can then extend the complex linear space $\mathfrak{T}_{\mathbb{C}}$ by adding the adjoint transformations and taking the norm-closure. We will denote such extension with the same symbol $\mathfrak{T}_{\mathbb{C}}$, which is now a C^*-algebra. Indeed, upon reconstructing $\mathfrak{E}_{\mathbb{C}}$ and $\mathfrak{T}_{\mathbb{C}}$ from the original real spaces via the Cartesian decomposition $\mathfrak{E}_{\mathbb{C}} = \mathfrak{E}_{\mathbb{R}} \oplus i\mathfrak{E}_{\mathbb{R}}$ and $\mathfrak{T}_{\mathbb{C}} = \mathfrak{T}_{\mathbb{R}} \oplus i\mathfrak{T}_{\mathbb{R}}$, and introducing the scalar product on $\mathfrak{E}_{\mathbb{C}}$ as the sesquilinear extension of a real symmetric scalar product $(\cdot,\cdot)_{\mathbb{R}}$ over $\mathfrak{E}_{\mathbb{R}}$, the adjoint of a real element $\mathscr{A} \in \mathfrak{T}_{\mathbb{R}}$ is just

[26] For a convex set, an automorphism must send the set to itself keeping the convex structure, whence it must be a one-to-one map that is linear on the span of the convex set.

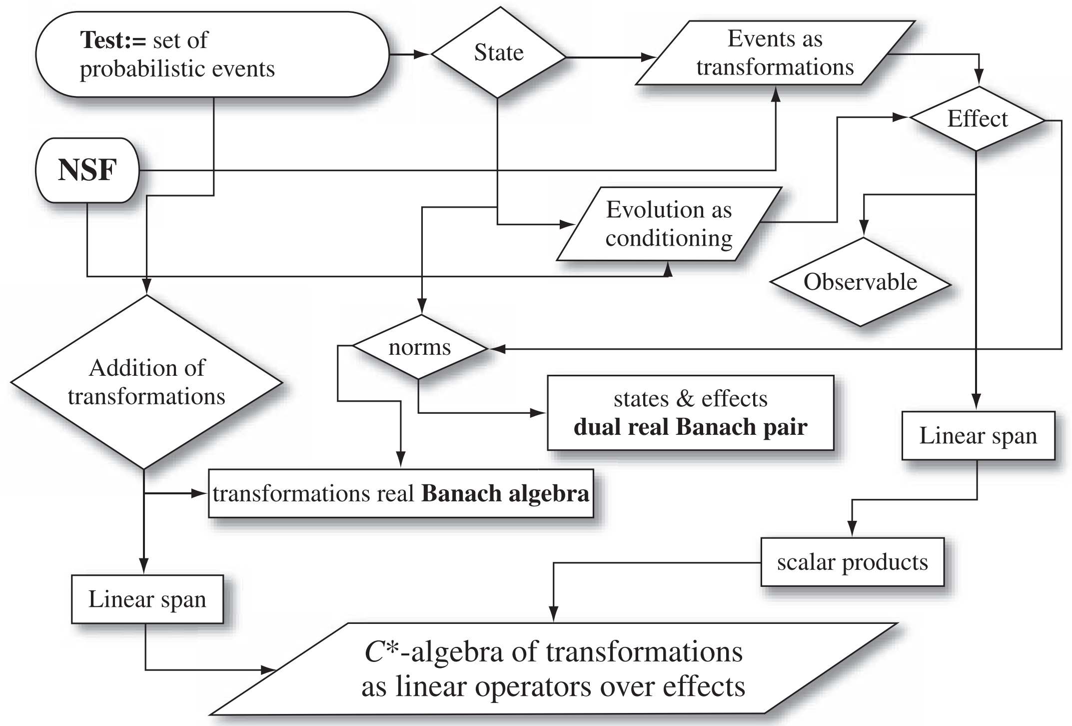


Fig. 5.1 A logical flow chart leading to the representation of any probabilistic theory in terms of a C^*-algebra of linear transformations over the linear space of complex effects (see also footnote 27 and Section 5.3.3 for an operational basis for the scalar product.)

the transposed matrix $\mathscr{A}^t$ with respect to a real basis orthonormal for $(\cdot,\cdot)_{\mathbb{R}}$, and $\mathscr{A}^\dagger := \mathscr{A}_R{}^t - \mathrm{i}\mathscr{A}_I{}^t$ for a general $\mathscr{A} = \mathscr{A}_R + \mathrm{i}\mathscr{A}_I \in \mathfrak{T}_{\mathbb{C}}$. A natural choice of matrix representation for $\mathfrak{T}_{\mathbb{R}}$ is given by its action over a minimal informational complete observable $\mathbb{L} = \{l_i\}$ (the scalar product $(\cdot,\cdot)_{\mathbb{R}} := (\cdot,\cdot)_{\mathbb{L}}$ will correspond to declaring $\mathbb{L}$ as orthonormal). Upon expanding $[l_i \circ \mathscr{A}]_{\mathrm{eff}}$ again over $\mathbb{L} = \{l_i\}$ one has the matrix representation $l_i \circ \mathscr{A} = \sum_j \mathscr{A}_{ji} l_j$. Using the fact that $\mathbb{L}$ is state-separating, we can write the probability rule as the pairing $\omega(a) = (\omega, a)_{\mathbb{R}}$ between $\mathfrak{E}_{\mathbb{R}}$ and $\mathfrak{S}_{\mathbb{R}}$ (and analogously for their complex spans).[27] In this way we see that for every probabilistic theory one can always represent transformations/events as elements of the C^*-algebra $\mathfrak{T}_{\mathbb{C}}$ of matrices acting on the linear space of complex effects $\mathfrak{E}_{\mathbb{C}}$. In Figure 5.1 the logical derivation of the C^*-algebra representation of the theory is summarized.

[27] The present derivation of the C^*-algebra representation of transformations is more direct than that in Ref. [17], and is just equivalent to the probabilistic framework inherent in the notion of a "test" (see also the summary of the whole logical deduction in the flow chart in Figure 5.1). The specific C^*-algebra in Ref. [17] possessed operational notions of adjoint and of scalar product over effects, both constructed using a symmetric faithful bipartite state, needing in this way two additional postulates: (a) the existence of dynamically independent systems and (b) the existence of faithful symmetric bipartite states. Such construction is briefly reviewed in Section 5.3.3.

Conversely, given (1) a C^*-algebra $\mathfrak{T}_\mathbb{C}$, (2) the cone of transformations $\mathfrak{T}_+$, and (3) the vector $e \in \mathfrak{E}_\mathbb{C}$ representing the deterministic effect, we can rebuild the full probabilistic theory by constructing the cone of effects as the orbit $\mathfrak{E}_+ = e \circ \mathfrak{T}_+$, and taking the cone of states $\mathfrak{S}_+$ as the dual cone of $\mathfrak{E}_+$.[28]

5.3 Independent systems

5.3.1 Dynamical independence and marginal states

A purely dynamical notion of *system independence* coincides with the possibility of performing local tests. To be precise, we will call systems S_1 and S_2 **independent** if it is possible to perform their tests as **local tests**, i.e., in such a way that for every joint state of S_1 and S_2 the transformations on S_1 commute with transformations on S_2, namely[29]

$$\mathscr{A}^{(1)} \circ \mathscr{B}^{(2)} = \mathscr{B}^{(2)} \circ \mathscr{A}^{(1)}, \ \forall \mathscr{A}^{(1)} \in \mathbb{A}^{(1)}, \ \forall \mathscr{B}^{(2)} \in \mathbb{B}^{(2)}. \tag{5.14}$$

The local tests comprise the Cartesian product $\mathsf{S}_1 \times \mathsf{S}_2$, which is closed under cascade. We will close this set also under convex combination, coarse-graining, and conditioning, making it a "system," denote such a system with the same symbol $\mathsf{S}_1 \times \mathsf{S}_2$, and call **local** all tests in $\mathsf{S}_1 \times \mathsf{S}_2$. We now **compose** the two systems S_1 and S_2 into the **bipartite** system $\mathsf{S}_1 \odot \mathsf{S}_2$ by adding the local tests into the new system $\mathsf{S}_1 \odot \mathsf{S}_2$ as $\mathsf{S}_1 \odot \mathsf{S}_2 \supseteq \mathsf{S}_1 \times \mathsf{S}_2$ and closing under cascading, coarse-graining, and convex combination. We call the tests in $\mathsf{S}_1 \odot \mathsf{S}_2 \backslash \mathsf{S}_1 \times \mathsf{S}_2$ **non-local**, and we will extend the local/non-local nomenclature to the pertaining transformations. In the following for identical systems we will also use the notation $\mathsf{S}^{\odot N} = \mathsf{S} \odot \mathsf{S} \odot \ldots \odot \mathsf{S}$ (N times), and $\mathfrak{Z}^{\odot N} := \mathfrak{Z}(\mathsf{S}^{\odot N})$ to denote N-partite sets/spaces, with $\mathfrak{Z} = \mathfrak{S}, \mathfrak{S}_+, \mathfrak{S}_\mathbb{R}, \mathfrak{S}_\mathbb{C}, \mathfrak{E}, \mathfrak{E}_+, \ldots$.

Since the local transformations commute, we will just put them in a string, as $(\mathscr{A}, \mathscr{B}, \mathscr{C}, \ldots) := \mathscr{A}^{(1)} \circ \mathscr{A}^{(2)} \circ \mathscr{A}^{(3)} \circ \ldots$ (convex combinations and coarse graining will be sums of strings). Clearly, since the probability $\omega(\mathscr{A}, \mathscr{B}, \mathscr{C}, \ldots)$ is independent of the time ordering of transformations, it is just a function only of the effects $\omega(\mathscr{A}, \mathscr{B}, \mathscr{C}, \ldots) = \omega([\mathscr{A}]_{\text{eff}}, [\mathscr{B}]_{\text{eff}}, [\mathscr{C}]_{\text{eff}}, \ldots)$, namely the joint effect corresponding to local transformations is made of (sums of) local effects $[(\mathscr{A}, \mathscr{B}, \mathscr{C}, \ldots)]_{\text{eff}} \equiv ([\mathscr{A}]_{\text{eff}}, [\mathscr{B}]_{\text{eff}}, [\mathscr{C}]_{\text{eff}}, \ldots)$.

The embedding of local tests $\mathsf{S}_1 \times \mathsf{S}_2$ into the bipartite system $\mathsf{S}_1 \odot \mathsf{S}_2$ implies that $\mathfrak{T}_\mathbb{F}(\mathsf{S}_1 \odot \mathsf{S}_2) \supseteq \mathfrak{T}_\mathbb{F}(\mathsf{S}_1) \otimes \mathfrak{T}_\mathbb{F}(\mathsf{S}_2)$ and $\mathfrak{E}_\mathbb{F}(\mathsf{S}_1 \odot \mathsf{S}_2) \supseteq \mathfrak{E}_\mathbb{F}(\mathsf{S}_1) \otimes \mathfrak{E}_\mathbb{F}(\mathsf{S}_2)$,

[28] The "orbit" $e \circ \mathfrak{T}_+$ is defined as the set $e \circ \mathfrak{T}_+ := \{e \circ \mathscr{A} | \mathscr{A} \in \mathfrak{T}_+\}$.

[29] The present definition of independent systems is purely dynamical, in the sense that it does not involve statistical requirements, e.g., the existence of factorized states. This, however, is implied by the mentioned no-restriction hypothesis for states.

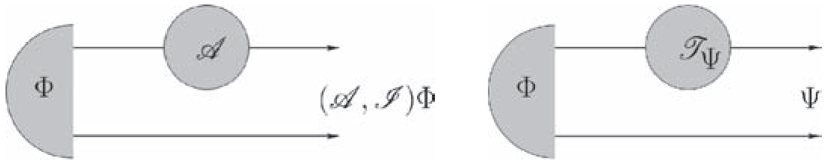


Fig. 5.2 Illustrations of the notions of a dynamically (left) and a preparationally (right) faithful state for a bipartite system. A bipartite state Φ is dynamically faithful with respect to system S_1 when the output state $(\mathscr{A}, \mathscr{I})\Phi$ is in one-to-one correspondence with the local transformation $\mathscr{A}$ on system S_1, whereas it is preparationally faithful with respect to S_1 if every bipartite state Ψ can be achieved as $\Psi = (\mathscr{T}_\Psi, \mathscr{I})\Phi$ via a local transformation $\mathscr{T}_\Psi$ on S_1.

both for real and for complex spans $\mathbb{F} = \mathbb{R}, \mathbb{C}$. On the other hand, since local tests include local state-preparation (or, otherwise, because of the no-restriction hypothesis for states) the set of bipartite states $\mathfrak{S}(\mathsf{S}_1 \odot \mathsf{S}_2)$ always includes the **factorized states**, i.e., those corresponding to factorized probability rules, e.g., $\Omega(a, b) = \omega_1(a)\omega_2(b)$ for local effects a and b. In parallel with local transformations and effects, we will denote factorized states as strings $\Omega = (\omega_1, \omega_2, \ldots)$, e.g., $(\omega_1, \omega_2)(a, b) = \omega_1(a)\omega_2(b)$. Then, closure under convex combination implies that $\mathfrak{S}_\mathbb{F}(\mathsf{S}_1 \odot \mathsf{S}_2) \supseteq \mathfrak{S}_\mathbb{F}(\mathsf{S}_1) \otimes \mathfrak{S}_\mathbb{F}(\mathsf{S}_2)$, for $\mathbb{F} = \mathbb{R}, \mathbb{C}$.

For N systems in the joint state Ω, we define the **marginal state** $\Omega|_n$ of the nth system as the probability rule for any local transformation $\mathscr{A}$ at the nth system, with all other systems untouched, namely

$$\Omega|_n(\mathscr{A}) \doteq \Omega(\mathscr{I}, \ldots, \mathscr{I}, \underbrace{\mathscr{A}}_{n\text{th}}, \mathscr{I}, \ldots). \tag{5.15}$$

Clearly, since the probability for local transformations depends only on their respective effects, the marginal state is equivalently defined as

$$\Omega|_n(a) \doteq \Omega(e, \ldots, e, \underbrace{a}_{n\text{th}}, e, \ldots) \quad \text{for } a \in \mathfrak{E}. \tag{5.16}$$

It readily follows that the marginal state $\Omega|_n$ is independent of any deterministic transformation – i.e., any test – that is performed on systems different from the nth: this is exactly the general statement of the **no-signaling** condition or **acausality of local tests**. Therefore, the present notion of dynamical independence directly implies the no-signaling condition. The definition in (5.15) can be trivially extended to unnormalized states.[30,31]

30 Notice that any generally unnormalized state is zero iff the joint state is zero, since $\Omega(e, e, \ldots, e) = \Omega_n(e) = 0$.

31 The present notion of dynamical independence is indeed so minimal that it can be satisfied not only by the quantum tensor product, but also by the quantum direct sum [36]. (Notice, however, that an analogue of Tsirelson's theorem [37] for transformations in finite dimensions would imply a representation of dynamical independence over the tensor product of effects.) In order to extract only the tensor product an additional assumption is needed. As shown in Refs. [17, 36] two possibilities are either postulating the existence of

In the following we will use the following identities:

$$\Psi|_2(a) = \Psi(e, a) = \Psi(e, e \circ \mathscr{A}) = (\mathscr{I}, \mathscr{A})\Psi(e, e), \ \forall \mathscr{A} \in a. \tag{5.17}$$

5.3.2 Faithful states

A bipartite state $\Phi \in \mathfrak{S}(\mathsf{S}_1 \odot \mathsf{S}_2)$ is **dynamically faithful** with respect to S_1 when the output state $(\mathscr{A}, \mathscr{I})\Phi$ is in one-to-one correspondence with the local transformation $\mathscr{A}$ on system S_1, that is, the cone-homomorphism[32] $\mathscr{A} \leftrightarrow (\mathscr{A}, \mathscr{I})\Phi$ from $\mathfrak{T}_+(\mathsf{S}_1)$ to $\mathfrak{S}_+(\mathsf{S}_1 \odot \mathsf{S}_2)$ is a monomorphism.[33] Equivalently the map $\mathscr{A} \mapsto (\mathscr{A}, \mathscr{I})\Phi$ extends to an injective linear map between the linear spaces $\mathfrak{T}_{\mathbb{R}}(\mathsf{S}_1)$ and $\mathfrak{S}_{\mathbb{R}}(\mathsf{S}_1 \odot \mathsf{S}_2)$ preserving the partial ordering relative to the spanning cones, and this is true also in the inverse direction on the range of the map. Notice that no physical transformation $\mathscr{A} \neq 0$ "annihilates" Φ, i.e., gives $(\mathscr{A}, \mathscr{I})\Phi = 0$.

A bipartite state $\Phi \in \mathfrak{S}(\mathsf{S}_1 \odot \mathsf{S}_2)$ is called **preparationally faithful** with respect to S_1 if every bipartite state Ψ can be achieved as $\Psi = (\mathscr{T}_\Psi, \mathscr{I})\Phi$ by a local transformation $\mathscr{T}_\Psi \in \mathfrak{T}_+(\mathsf{S}_1)$. This means that the cone-homomorphism $\mathscr{A} \mapsto (\mathscr{A}, \mathscr{I})\Phi$ from $\mathfrak{T}_+(\mathsf{S}_1)$ to $\mathfrak{S}_+(\mathsf{S}_1 \odot \mathsf{S}_2)$ is an epimorphism. Equivalently, the map $\mathscr{A} \mapsto (\mathscr{A}, \mathscr{I})\Phi$ extends to a surjective linear map between the linear spaces $\mathfrak{T}_{\mathbb{R}}(\mathsf{S}_1)$ and $\mathfrak{S}_{\mathbb{R}}(\mathsf{S}_1 \odot \mathsf{S}_2)$ preserving the partial ordering relative to the spanning cones.

In simple words, a dynamically faithful state keeps the imprinting of a local transformation on the output, i.e., from the output we can recover the transformation. On the other hand, a preparationally faithful state allows us to prepare any desired joint state (probabilistically) by means of local transformations. Dynamical and preparational faithfulness correspond to the properties of being *separating* and *cyclic* for the C^*-algebra of transformations.

Theorem 2. *The following assertions hold.*

(1) *Any state $\Phi \in \mathfrak{S}(\mathsf{S}_1 \odot \mathsf{S}_2)$ that is preparationally faithful with respect to S_1 is dynamically faithful with respect to S_2.*
(2) *For identical systems in finite dimensions any state Φ that is preparationally faithful with respect to a system is also dynamically faithful with respect to the same system,*

bipartite states that are dynamically and preparationally faithful, or postulating the local observability principle. Here we will consider the former as a postulate, and derive the latter as a theorem.

[32] A cone-homomorphism between cones C_1 and C_2 is a linear map between $\mathsf{Span}_{\mathbb{R}}(\mathsf{C}_1)$ and $\mathsf{Span}_{\mathbb{R}}(\mathsf{C}_2)$ that sends elements of C_1 to elements of C_2, but not necessarily vice versa.

[33] This means that $(\mathscr{A}_1, \mathscr{I})\Phi = (\mathscr{A}_2, \mathscr{I})\Phi$ iff $\mathscr{A}_1 = \mathscr{A}_2$, or, in other words, $\forall \mathscr{A} \in \mathfrak{T}_{\mathbb{R}}$: $(\mathscr{A}, \mathscr{I})\Phi = 0 \iff \mathscr{A} = 0$.

and one has the cone-isomorphism[34] $\mathfrak{T}_+(\mathsf{S}) \simeq \mathfrak{S}_+(\mathsf{S}^{\odot 2})$. *Moreover, a local transformation on* Φ *produces an output pure (unnormalized) bipartite state iff the transformation is atomic.*

(3) *If there exists a state of* $\mathsf{S}_1 \odot \mathsf{S}_2$ *that is preparationally faithful with respect to* S_1*, then* $\dim(\mathsf{S}_1) \geqslant \dim(\mathsf{S}_2)$.

(4) *If there exists a state of* $\mathsf{S}_1 \odot \mathsf{S}_2$ *that is preparationally faithful with respect to both systems, then one has the cone-isomorphisms* $\mathfrak{E}_+(\mathsf{S}_1) \simeq \mathfrak{S}_+(\mathsf{S}_2)$ *and* $\mathfrak{E}_+(\mathsf{S}_2) \simeq \mathfrak{S}_+(\mathsf{S}_1)$.

(5) *If for two identical systems there exists a state that is preparationally faithful with respect to both systems, then one has the cone-isomorphism* $\mathfrak{S}_+ \simeq \mathfrak{E}_+$ *(weak self-duality).*

(6) *If the state* $\Phi \in \mathfrak{S}(\mathsf{S}_1 \odot \mathsf{S}_2)$ *is preparationally faithful with respect to* S_1*, then for any invertible transformation* $\mathscr{A} \in \mathfrak{T}_+(\mathsf{S}_1)$ *also the (unnormalized) state* $(\mathscr{A}, \mathscr{I})\Phi$ *is preparationally faithful with respect to the same system. In particular, it will be a faithful state for any physical automorphism of* $\mathfrak{S}(\mathsf{S}_1)$.[35]

(7) *For identical systems in finite dimensions, for* Φ *preparationally faithful with respect to both systems, the state* $\chi := \Phi(e, \cdot)$ *is cyclic in* $\mathfrak{S}_+(\mathsf{S})$ *under* $\mathfrak{T}_+(\mathsf{S})$*, and the observables* $\mathbb{L} = \{l_i\}$ *of* S_2 *are in one-to-one correspondence with the ensemble decompositions* $\{\rho_i\}_{i=1}^{|\mathbb{L}|}$ *of* χ*, with* $\rho_i := \Phi(l_i, \cdot)$*, and* χ *is an internal state.*

Proof.

(1) Introduce the map $\omega \mapsto \mathscr{T}_\omega$ where for every $\omega \in \mathfrak{S}(\mathsf{S}_2)$ one chooses a local transformation $\mathscr{T}_\omega$ on S_1 such that $(\mathscr{T}_\omega, \mathscr{I})\Phi|_2 = \omega$. This is possible because Φ is preparationally faithful with respect to S_1. One has $\mathscr{A}\omega = (\mathscr{T}_\omega, \mathscr{A})\Phi|_2 = (\mathscr{T}_\omega, \mathscr{I})(\mathscr{I}, \mathscr{A})\Phi|_2 \ \forall \omega \in \mathfrak{S}(\mathsf{S}_2)$. Therefore, from $(\mathscr{I}, \mathscr{A})\Phi$ one can recover the action of $\mathscr{A}$ on any state ω by first applying $(\mathscr{T}_\omega, \mathscr{I})$ and then take the marginal, i.e., one recovers $\mathscr{A}$ from $(\mathscr{I}, \mathscr{A})\Phi$, which is another way of saying that $\mathscr{A} \mapsto (\mathscr{I}, \mathscr{A})\Phi$ is injective, namely Φ is dynamically faithful with respect to S_2.

(2) Denote by $\Phi \in \mathfrak{S}^{\odot 2}$ a state that is preparationally faithful with respect to S_1. Since the linear map $\mathscr{A} \mapsto (\mathscr{A}, \mathscr{I})\Phi$ from $\mathfrak{T}_\mathbb{R}$ to $\mathfrak{S}_\mathbb{R}^{\odot 2}$ is surjective, one has $\dim(\mathfrak{T}_\mathbb{R}) \geqslant \dim(\mathfrak{S}_\mathbb{R}^{\odot 2})$. However, one has also $\dim(\mathfrak{T}_\mathbb{R}) \leqslant \dim(\mathfrak{S}_\mathbb{R}^{\odot 2})$ since $\mathfrak{T}_\mathbb{R} \subseteq \mathsf{Lin}(\mathfrak{S}_\mathbb{R}) \simeq \mathfrak{S}_\mathbb{R}^{\otimes 2} \subseteq \mathfrak{S}_\mathbb{R}^{\odot 2}$, whence $\dim(\mathfrak{T}_\mathbb{R}) = \dim(\mathfrak{S}_\mathbb{R}^{\odot 2})$, and, having null kernel, the map is also injective, whence Φ is dynamically faithful with respect to S_1. Since now the state Φ is both preparationally and dynamically faithful with respect to the same system S_1, it

[34] We say that two cones C_1 and C_2 are isomorphic (denoted as $\mathsf{C}_1 \simeq \mathsf{C}_2$) if there exists a one-to-one linear mapping between $\mathsf{Span}_\mathbb{R}(\mathsf{C}_1)$ and $\mathsf{Span}_\mathbb{R}(\mathsf{C}_2)$ that is cone-preserving in both directions. We will call such a map a cone-isomorphism between the two cones. Such a map will send extremal rays of C_1 to extremal rays of C_2 and positive linear combinations to positive linear combinations, and the same is true for the inverse map.

[35] One may be tempted to consider all automorphisms of $\mathfrak{S}(\mathsf{S}_1)$, instead of just the physical ones. However, there is no guarantee that any automorphism will be also an automorphism of bipartite states when applied locally. This is the case of QM, where the transposition is an automorphism of $\mathfrak{S}(\mathsf{S}_1)$, and nevertheless is not a local automorphism of $\mathfrak{S}(\mathsf{S}_1 \odot \mathsf{S}_2)$.

follows that the map $\mathscr{A} \mapsto (\mathscr{A}, \mathscr{I})\Phi$ establishes the cone-isomorphism $\mathfrak{T}_+ \simeq \mathfrak{S}_+^{\odot 2}$. Since the faithful state establishes the cone-isomorphism $\mathfrak{T}_+ \simeq \mathfrak{S}_+^{\odot 2}$, it maps extremal rays of $\mathfrak{T}_+$ to extremal rays of $\mathfrak{S}_+^{\odot 2}$ and vice versa; that is, $\mathscr{A} \in \mathsf{Erays}(\mathfrak{T}_+)$ iff $(\mathscr{A}, \mathscr{I})\Phi \in \mathsf{Erays}(\mathfrak{S}_+^{\odot 2})$.

(3) For Φ preparationally faithful with respect to S_1, consider the cone homomorphism $a \mapsto \omega_a := \Phi(a, \cdot)$ which associates an (unnormalized) state $\omega_a \in \mathfrak{S}_+(\mathsf{S}_2)$ with each effect $a \in \mathfrak{E}_+(\mathsf{S}_1)$. The extension to a linear map $a \mapsto \omega_a$ between the linear spaces $\mathfrak{S}_{\mathbb{R}}(\mathsf{S}_2)$ and $\mathfrak{E}_{\mathbb{R}}(\mathsf{S}_1)$ preserves the cone structure, and is surjective, since Φ is preparationally faithful with respect to S_1 (whence every bipartite state, and, in particular, every marginal state, can be obtained from a local effect). The bound $\dim(\mathsf{S}_1) \geqslant \dim(\mathsf{S}_2)$ then follows from surjectivity.

(4) Similarly to the proof of item (1), consider the map $\lambda \mapsto \mathscr{T}_\lambda$, where for every marginal state $\lambda \in \mathfrak{S}(\mathsf{S}_1)$ one chooses a local transformation $\mathscr{T}_\lambda$ on S_2 such that $(\mathscr{I}, \mathscr{T}_\lambda)\Phi|_1 = \lambda$ (Φ is preparationally faithful with respect to S_2). Then, one has

$$\forall \lambda \in \mathfrak{S}(\mathsf{S}_1),\ \lambda(a) = (\mathscr{I}, \mathscr{T}_\lambda)\Phi(a, e) = \Phi(a, \mathscr{T}_\lambda) = \omega_a(\mathscr{T}_\lambda). \tag{5.18}$$

It follows that $\omega_a = \omega_b$ implies that $\lambda(a) = \lambda(b)$ for all states $\lambda \in \mathfrak{S}(\mathsf{S}_1)$; that is, $a = b$, whence the homomorphism $a \mapsto \omega_a$ which is surjective (since Φ is preparationally faithful) is also injective, i.e., is bijective, and, since it maps elements of $\mathfrak{E}_+(\mathsf{S}_1)$ to elements of $\mathfrak{S}_+(\mathsf{S}_2)$ and, vice versa, to each element of $\mathfrak{S}_+(\mathsf{S}_2)$, it corresponds to an element of $\mathfrak{E}_+(\mathsf{S}_1)$ (Φ is preparationally faithful), thus it is a cone-isomorphism. We then have the cone-isomorphism $\mathfrak{E}_+(\mathsf{S}_1) \simeq \mathfrak{S}_+(\mathsf{S}_2)$. The cone-isomorphism $\mathfrak{E}_+(\mathsf{S}_2) \simeq \mathfrak{S}_+(\mathsf{S}_1)$ follows on exchanging the two systems.

(5) According to point (4) one has the cone-isomorphism $\mathfrak{E}_+(\mathsf{S}_1) \simeq \mathfrak{S}_+(\mathsf{S}_2) \simeq \mathfrak{S}_+(\mathsf{S}_1)$.

(6) This is obvious, from the definition of a preparationally faithful state.

(7) According to (4) $\omega_a := \Phi(a, \cdot)$ establishes the cone-isomorphism $\mathfrak{E}_+(\mathsf{S}) \simeq \mathfrak{S}_+(\mathsf{S})$. On the other hand, since the state is both preparationally and dynamically faithful for either system, then for any transformation $\mathscr{T}$ on the first system there exists a unique transformation $\mathscr{T}'$ on the other system giving the same output state (see also the definition of the "transposed" transformation with respect to a dynamically faithful state in the following). Therefore, since any effect a can be written as $a = e \circ \mathscr{T}_a$ for any $\mathscr{T}_a \in a$, one has $\omega_a = \Phi(e \circ \mathscr{T}_a, \cdot) = \Phi(e, \cdot \circ \mathscr{T}'_a) = \mathscr{T}'_a \chi$. The observable–ensemble correspondence and the fact that χ is an internal state are both immediate consequences of the fact that $\omega_a := \Phi(a, \cdot)$ is a cone-isomorphism.

The transposed of a transformation (Figure 5.3). For a symmetric bipartite state Φ of two identical systems that is preparationally faithful for one system – hence, according to Theorem 2, is both dynamically and preparationally faithful with respect to both systems – one can define operationally the **transposed** $\mathscr{T}'$ of a transformation $\mathscr{T} \in \mathfrak{T}_{\mathbb{R}}$ through the identity

$$\Phi(a, b \circ \mathscr{T}) = \Phi(a \circ \mathscr{T}', b), \tag{5.19}$$

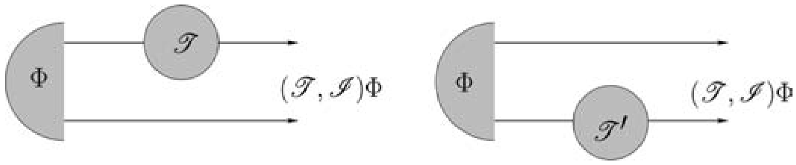


Fig. 5.3 An illustration of the notion of the transposed of a transformation for a symmetric dynamically and preparationally faithful state.

i.e., $(\mathscr{T}', \mathscr{I})\Phi = (\mathscr{I}, \mathscr{T})\Phi$, namely, operationally the transposed $\mathscr{T}'$ of a transformation $\mathscr{T}$ is the transformation which will give the same output bipartite state of $\mathscr{T}$ if operated on the twin system. It is easy to verify (using the symmetry of Φ) that $\mathscr{T}'' = \mathscr{T}$ and that $(\mathscr{B} \circ \mathscr{A})' = \mathscr{A}' \circ \mathscr{B}'$.

We are now in position to formulate the main postulate.

Postulate PFAITH (Existence of a symmetric preparationally faithful pure state). *For any couple of identical systems, there exists a symmetric (under permutation of the two systems) pure state that is preparationally faithful.*

Theorem 2 guarantees that such a state is both dynamically and preparationally faithful, and with respect to both systems, as a consequence of symmetry.[36] Postulate PFAITH thus guarantees that to any system we can adjoin an ancilla and prepare a pure state that is dynamically and preparationally faithful with respect to our system. This is operationally crucial in guaranteeing the preparability of any quantum state for any bipartite system using only local transformations, and to assure the possibility of experimental calibrability of tests for any system. Notice that it would be impossible, even in principle, to calibrate transformations without a dynamically faithful state, since any set of input states $\{\omega_n\} \in \mathsf{S}'$ that is "separating" for transformations $\mathfrak{T}(\mathsf{S}')$ is equivalent to a bipartite state $\Phi = \sum_n \omega_n \otimes \lambda_n \in \mathfrak{S}(\mathsf{S}' \odot \mathsf{S}'')$ that is dynamically faithful for S', with the states $\{\lambda_n\}$ working just as "flags" representing the "knowledge" of which state of the set $\{\omega_n\}$ has been prepared. Notice that in QM every maximal Schmidt-number entangled state of two identical systems is both preparationally and dynamically faithful for both systems. In classical mechanics, on the other hand, a state of the form $\Phi = \sum_l |l\rangle\langle l| \otimes |l\rangle\langle l|$ with $\{|l\rangle\}$ a complete orthogonal set of states (see footnote 19) will be both dynamically and preparationally faithful; however, being not pure, it would require a (possibly unlimited) sequence of preparations.

On the mathematical side, instead, according to Theorem 2 Postulate PFAITH restricts the theory to the weakly self-dual scenario (i.e., with the

[36] In fact, upon denoting by $\mathscr{T}_\Psi$ the local transformation such that $(\mathscr{T}, \mathscr{I})\Phi = \Psi$, one has $(\mathscr{I}, \mathscr{T}_{\mathscr{S}\Psi})\Phi = \Psi$, $\mathscr{S}$ denoting the transformation swapping the two systems.

cone-isomorphism $\mathfrak{S}_+ \simeq \mathfrak{E}_+$), and in finite dimensions one also has the cone-isomorphism $\mathfrak{T}_+(\mathsf{S}) \simeq \mathfrak{S}_+(\mathsf{S}^{\odot 2})$. In addition, one also has the following very useful lemma.

Lemma 5. *For finite dimensions Postulate PFAITH implies that the linear space of transformations is full, i.e.,* $\mathfrak{T}_{\mathbb{F}} = \mathsf{Lin}(\mathfrak{E}_{\mathbb{F}})$. *Moreover, one has* $\mathfrak{S}_{\mathbb{F}}(\mathsf{S}^{\odot 2}) = \mathfrak{S}_{\mathbb{F}}(\mathsf{S})^{\otimes 2}$ *and* $\mathfrak{E}_{\mathbb{F}}(\mathsf{S}^{\odot 2}) = \mathfrak{E}_{\mathbb{F}}(\mathsf{S})^{\otimes 2}$ *for* $\mathbb{F} = \mathbb{R}, \mathbb{C}$, *that is, bipartite states and effects are cones spanning the tensor products* $\mathfrak{S}_{\mathbb{F}}^{\otimes 2}$ *and* $\mathfrak{E}_{\mathbb{F}}^{\otimes 2}$, *respectively.*

Proof. In the following we restrict to finite dimensions, with $\mathbb{F} = \mathbb{R}, \mathbb{C}$ denoting either the real or the complex fields, respectively. According to item (2) of Theorem 2, for two identical systems the existence of a state that is preparationally faithful with respect to either one of the two systems implies $\mathfrak{S}_{\mathbb{F}}(\mathsf{S}^{\odot 2}) \simeq \mathfrak{T}_{\mathbb{F}}(\mathsf{S})$. Since transformations act linearly over effects, one has $\mathfrak{T}_{\mathbb{F}} \subseteq \mathsf{Lin}(\mathfrak{E}_{\mathbb{F}}) \simeq \mathfrak{E}_{\mathbb{F}}^{\otimes 2}$, whence $\mathfrak{E}_{\mathbb{F}}(\mathsf{S}^{\odot 2}) \simeq \mathfrak{S}_{\mathbb{F}}(\mathsf{S}^{\odot 2}) \simeq \mathfrak{T}_{\mathbb{F}}(\mathsf{S}) \subseteq \mathfrak{E}_{\mathbb{F}}(\mathsf{S})^{\otimes 2}$. However, by local-test embedding one also has $\mathfrak{E}_{\mathbb{F}}(\mathsf{S}^{\odot 2}) \supseteq \mathfrak{E}_{\mathbb{F}}(\mathsf{S})^{\otimes 2}$, whence $\mathfrak{E}_{\mathbb{F}}(\mathsf{S}^{\odot 2}) = \mathfrak{E}_{\mathbb{F}}(\mathsf{S})^{\otimes 2}$, which implies that $\mathfrak{T}_{\mathbb{F}} = \mathsf{Lin}(\mathfrak{E}_{\mathbb{F}})$. Finally, by virtue of state-effect duality one also has $\mathfrak{S}_{\mathbb{F}}(\mathsf{S}^{\odot 2}) = \mathfrak{S}_{\mathbb{F}}^{\otimes 2}(\mathsf{S})$.

The above lemma could have been extended to couples of different systems. However, this would necessitate the consideration of more general transformations between different systems (see footnote 13).

We conclude that Postulate PFAITH – i.e., the existence of a symmetric preparationally faithful pure state for bipartite systems – guarantees that we can represent bipartite quantities (states, effects, transformations) as elements of the tensor product of the single-system spaces. This fact also implies the following relevant principle.

Corollary 2. (Local observability principle). *For every composite system there exist informationally complete observables made of local informationally complete observables.*

Proof. A joint observable made of local observables $\mathbb{L} = \{l_i\}$ on S_1 and $\mathbb{M} = \{m_j\}$ on S_2 is of the form $\mathbb{L} \times \mathbb{M} = \{(l_i, m_j)\}$. Then, by definition, the statement of the corollary is $\mathfrak{E}_{\mathbb{R}}(\mathsf{S}^{\odot 2}) \subseteq \mathsf{Span}_{\mathbb{R}}(\mathbb{L} \times \mathbb{M}) = \mathfrak{E}_{\mathbb{R}}^{\otimes 2}(\mathsf{S})$, which is true according to Lemma 5.

Operationally, the local-observability principle plays a crucial role, since it reduces enormously experimental complexity, by guaranteeing that only local (although jointly executed) tests are sufficient to retrieve complete information on a composite system, including all correlations between the components. This principle reconciles holism with reductionism in a non-local theory, in the sense that we can observe a holistic nature in a reductionistic way, i.e., locally.

In addition to Lemma 5 and to the local-observability principle, Postulate PFAITH has a long list of remarkable consequences for the probabilistic theory, which are given by the following theorem.

Theorem 3. *If PFAITH holds, the following assertions are true.*

(1) *The identity transformation is atomic.*
(2) *One has $\omega_{a\circ\mathscr{A}'} = \mathscr{A}\omega_a$, or equivalently $\mathscr{A}\omega = \Phi(a_\omega \circ \mathscr{A}', \cdot)$, where $\mathscr{A}'$ denotes the transposed of $\mathscr{A}$ with respect to Φ.*
(3) *The transposed of a physical automorphism of the set of states is still a physical automorphism of the set of states.*
(4) *The marginal state χ is invariant under the transposed of a channel (deterministic transformation) and hence, in particular, under a physical automorphism of the set of states.*
(5) *Alice can perform perfect EPR-cheating in a perfect concealing bit-commitment protocol.*

Proof.

(1) According to Theorem 2 item (2), the map $\mathscr{A} \mapsto (\mathscr{A}, \mathscr{I})\Phi$ establishes the cone-isomorphism $\mathfrak{T}_+ \simeq \mathfrak{S}_+^{\odot 2}$, whence on mapping extremal rays of $\mathfrak{T}_+$ to extremal rays of $\mathfrak{S}_+^{\odot 2}$ and vice versa it maps the state Φ itself (which is pure) to the identity, which then must be atomic.
(2) Immediate definition of the transposition with respect to the dynamically faithful state Φ.
(3) Point (2) establishes that the transposed of a state-automorphism is an effect automorphism, which, due to the cone-isomorphism, is again a state-automorphism (see also footnote 35).
(4) For deterministic $\mathscr{T}$ one has $\mathscr{T}'\chi = \Phi(e, \cdot \circ \mathscr{T}') = \Phi(e \cdot \mathscr{T}, \cdot) = \Phi(e, \cdot) = \chi$. The last statement follows from (3) (see also footnote 35).
(5) (For the definition of the protocol, see Ref. [38]). For the protocol to be concealing there must exist two ensembles of states $\{\rho_i^{\mathbb{A}}\}$ and $\{\rho_i^{\mathbb{B}}\}$ that are indistinguishable by Bob. For $\sum_i \rho_i^{\mathbb{A}} = \sum_i \rho_i^{\mathbb{B}} = \chi$ these correspond to the two observables $\mathbb{A} = \{a_i\}$ and $\mathbb{B} = \{b_i\}$ with $\rho_i^{\mathbb{A}} = \Phi(a_i, \cdot)$ and $\rho_i^{\mathbb{B}} = \Phi(b_i, \cdot)$. Instead of sending to Bob a state from either one of the two ensembles, Alice can cheat by “entangling” her ancilla (system S_1) with Bob’s system in the state Φ, and then measuring either one of the observables $\mathbb{A} = \{a_i\}$ and $\mathbb{B} = \{b_i\}$.

Notice that atomicity of identity occurs in QM, whereas it is not true in a classical probabilistic theory (see footnote 19). In classical mechanics one can gain information on the state without making a disturbance thanks to the non-atomicity of the identity transformation. According to Theorem 3 item (1) the need of disturbance for gaining information is a consequence of the purity of the preparationally faithful state, whence disturbance is the price to be paid for the reduction of the preparation complexity.

5.3.3 The Scalar product over effects induced by a symmetric faithful state

In this subsection I briefly review the construction in Ref. [17] of a scalar product over $\mathfrak{E}_{\mathbb{C}}$ via a symmetric faithful state, together with the corresponding operational definition of "transposed" and "complex conjugation" – with the composition of the two giving the adjoint.

According to Theorem 2 item (2), for two identical systems in finite dimensions any state that is preparationally faithful with respect to a system is also dynamically faithful with respect to the same system. Moreover, according to Postulate PFAITH, there always exists such a state, say Φ, which is symmetric under permutation of the two systems. The state Φ is then a symmetric real form over $\mathfrak{E}_{\mathbb{R}}$, whence it provides a non-degenerate scalar product over $\mathfrak{E}_{\mathbb{R}}$ via its Jordan form

$$\forall a, b \in \mathfrak{E}_{\mathbb{R}},\ {}_{\Phi}(b|a)_{\Phi} := |\Phi|(b, a) = \Phi(\varsigma(b), a), \tag{5.20}$$

where ς is the involution $\varsigma = \pi_+ - \pi_-$, $\pi_\pm$ denoting the orthogonal projectors over the positive (negative) eigenspaces of the symmetric form, or, explicitly, $\varsigma(a) := \sum_j \Phi(a, \tilde{f}_j)\tilde{f}_j$ and $\{\tilde{f}_j\}$ is the canonical Jordan basis.[37] Notice that the Jordan form is representation-dependent – i.e., it is defined through the reference test $\mathbb{L} = \{l_i\}$ – whereas its signature – i.e., the difference between the numbers of positive and negative eigenvalues – will be a property of the system S, and will generally depend on the specific probabilistic theory. For transformations $\mathscr{T} \in \mathfrak{T}_{\mathbb{R}}$ we define $a \circ \varsigma(\mathscr{T}) := \varsigma(\varsigma(a) \circ \mathscr{T}) =: a \circ \mathscr{Z} \circ \mathscr{T} \circ \mathscr{Z}$. For the identity transformation we have $\varsigma(\mathscr{I}) = \mathscr{Z} \circ \mathscr{Z} = \mathscr{I}$. Corresponding to a symmetric faithful bipartite state Φ one has the generalized transformation $\mathscr{T}_\Phi$, given by

$$a \circ \mathscr{T}_\Phi := \sum_k \Phi(l_k, a) l_k, \tag{5.21}$$

for a fixed orthonormal basis $\mathbb{L} = \{l_j\}$, and in terms of the corresponding symmetric scalar product $(\cdot,\cdot)_{\mathbb{L}}$ introduced in Section 5.2.10, one has

$$(a, b \circ \mathscr{T}_\Phi)_{\mathbb{L}} = (a \circ \mathscr{T}_\Phi, b)_{\mathbb{L}} = \Phi(a, b). \tag{5.22}$$

Using the dynamical and preparational faithfulness of Φ we have defined operationally the transposed $\mathscr{T}'$ of a transformation $\mathscr{T} \in \mathfrak{T}_{\mathbb{R}}$. Such an "operational" transposed is related to the transposed $\tilde{\mathscr{C}}$ under the scalar product $(\cdot,\cdot)_{\mathbb{L}}$ as $\mathscr{C}' = \mathscr{T}_\Phi \circ \tilde{\mathscr{C}} \circ \mathscr{T}_\Phi^{-1}$. It is easy to check that $\tilde{\mathscr{Z}} = \mathscr{Z} = \mathscr{Z}'$.

On the complex linear span $\mathfrak{T}_{\mathbb{C}}$ one can introduce a scalar product as the sesquilinear extension of the real symmetric scalar product $(\cdot,\cdot)_\Phi$ over $\mathfrak{E}_{\mathbb{R}}$ via

[37] In the diagonalizing orthonormal basis one has $s_j\delta_{ij} = \Phi(\tilde{f}_i, \tilde{f}_j) = |\lambda_j|^{-1}\Phi(f_i, f_j)$, $s_j = \pm 1$, $\tilde{f}_j = f_j/\sqrt{|\lambda_j|}$.

the complex conjugation $\eta(\mathscr{T}) = \mathscr{T}_{\mathrm{R}} - \mathrm{i}\mathscr{T}_{\mathrm{I}}$, $\mathscr{T}_{\mathrm{R,I}} \in \mathfrak{T}_{\mathbb{R}}$, and the adjoint for the sesquilinear scalar product is then given by

$$\mathscr{T}^{\dagger} = \mathscr{Z} \circ \eta(\mathscr{T}') \circ \mathscr{Z} = |\mathscr{T}_{\Phi}| \circ \eta(\tilde{\mathscr{T}}) \circ |\mathscr{T}_{\Phi}|^{-1}, \tag{5.23}$$

namely $\mathscr{T}^{\dagger} = \mathscr{Z} \circ \mathscr{T}' \circ \mathscr{Z}$ on real transformations $\mathscr{T} \in \mathfrak{T}_{\mathbb{R}}$. The Jordan involution ς thus plays the role of a complex conjugation on $\mathfrak{T}_{\mathbb{R}}$, which must be anti-linearly extended to $\mathfrak{T}_{\mathbb{C}}$.

The faithful state Φ becomes a cyclic and separating vector of a GNS representation on noticing that $(\mathscr{A}^{(2)}\Phi)(\eta\varsigma b, a) = {}_{\Phi}(b, a \circ \mathscr{A})_{\Phi}$,[38] and in (5.23) one can recognize the Tomita–Takesaki modular operator of the representation [39].

5.4 Axiomatic interlude: exploring Postulates FAITHE and PURIFY

In this section we investigate two additional postulates of a probabilistic theory: Postulate FAITHE – the existence of a faithful effect (somehow dual to Postulate PFAITH) – and Postulate PURIFY – the existence of a purification for every state. As we will see, these new postulates bring the probabilistic theory closer and closer to QM. However, I was still unable to prove (or to find counterexamples) that with these two additional postulates the probabilistic theory *is* QM.

5.4.1 FAITHE: a postulate on a faithful effect

As previously mentioned, Postulate FAITHE is somehow the dual version of Postulate PFAITH.[39]

Postulate FAITHE (Existence of a faithful effect). *There exists a bipartite effect* $F \in \mathfrak{E}(\mathsf{S}^{\odot 2})$ *achieving the inverse of the isomorphism* $a \mapsto \omega_a := \Phi(a, \cdot)$. *More precisely,*

[38] The action of the algebra of generalized transformations on the first system corresponds to the transposed representation $(\mathscr{A}^{(1)}\Phi)(\eta\varsigma b, a) = \Phi(\eta\varsigma b \circ \mathscr{A}, a) = \Phi(\eta\varsigma b, a \circ \mathscr{A}') = (\mathscr{A}'^{(2)}\Phi)(\eta\varsigma b, a)$.

[39] At first sight it seems that the existence of an effect F such that $F_{23}\Phi_{12}\Phi_{34} = \alpha\Phi_{14}$ could be derived directly from PFAITH. Indeed, according to Lemma 5 for finite dimensions and identical systems we have $\mathfrak{S}_{\mathbb{F}}(\mathsf{S}^{\odot 2}) = \mathfrak{S}_{\mathbb{F}}(\mathsf{S})^{\otimes 2}$ and $\mathfrak{E}_{\mathbb{F}}(\mathsf{S}^{\odot 2}) = \mathfrak{E}_{\mathbb{F}}(\mathsf{S})^{\otimes 2}$ for $\mathbb{F} = \mathbb{R}, \mathbb{C}$. Moreover, according to Theorem 2 item (4) the map $a \mapsto \omega_a = \Phi(a, \cdot)$, for Φ symmetric preparationally faithful achieves the cone-isomorphism $\mathfrak{S}_{+} \simeq \mathfrak{E}_{+}$, whence for the bipartite system one has $\mathfrak{S}_{+}(\mathsf{S}^{\odot 2}) \simeq \mathfrak{E}_{+}(\mathsf{S}^{\odot 2})$. This leads one to think that it should be possible to achieve a preparationally faithful state for $\mathsf{S}^{\odot 4}$ as the product $\Phi_{12}\Phi_{34}$. However, this is not necessarily true. In fact, since the map $\mathfrak{E}_{\mathbb{F}}(\mathsf{S})^{\otimes 2} \ni E \mapsto \Omega_E = E_{23}\Phi_{12}\Phi_{34}$ is a linear bijection between $\mathfrak{E}_{\mathbb{F}}(\mathsf{S})^{\otimes 2}$ and $\mathfrak{S}_{\mathbb{F}}(\mathsf{S})^{\otimes 2}$ (since $\mathsf{Span}_{\mathbb{F}}\{\Phi_{12}(\cdot, a)\Phi_{34}(b, \cdot) | a, b \in \mathfrak{E}\} = \mathfrak{S}_{\mathbb{F}}(\mathsf{S})^{\otimes 2} = \mathfrak{S}_{\mathbb{F}}(\mathsf{S}^{\odot 2})$) is cone-preserving, it sends separable effects to separable states, whence it sends non-separable effects to non-separable states (since it is one-to-one). However, it doesn't necessarily achieve the cone-isomorphism $\mathfrak{S}_{+}(\mathsf{S}^{\odot 2}) \simeq \mathfrak{E}_{+}(\mathsf{S}^{\odot 2})$, since it is not necessarily true that any bipartite state Ω is the mapped of a bipartite effect E_{Ω} (we remember that a cone-isomorphism is a bijection that preserves the cone *in both directions*). If by chance this were the case – i.e., $E \mapsto \Omega_E$ is a cone-isomorphism for $\mathsf{S}^{\odot 2}$ – then this would mean that there exists an effect $F \in \mathfrak{E}(\mathsf{S}^{\odot 2})$ such that $\Omega_F = \alpha\Phi$, with $0 < \alpha \leqslant 1$.

$$F_{23}(\omega_a)_2 = F_{23}\Phi_{12}(a, \cdot) = \alpha a_3, \quad 0 < \alpha \leqslant 1. \tag{5.24}$$

Notice that, since Φ establishes an isomorphism between the cones of states and effects, there must exist a generalized effect $F \in \mathfrak{E}_{\mathbb{R}}^{\otimes 2}$ satisfying (5.24), but we are not guaranteed that it is a physical one, i.e., $F \in \mathfrak{E}_{+}(\mathsf{S}^{\odot 2})$.

Let's denote by $\hat{F} = \alpha^{-1}F$ the rescaled effect in the cone. Equation (5.24) can be rewritten in different notation as follows:

$$\hat{F}(\omega_a, \cdot) = \hat{F}(\Phi(a, \cdot), \cdot) = a, \tag{5.25}$$

$$\Phi(a_\omega, \cdot) = \Phi(\hat{F}(\omega, \cdot), \cdot) = \omega. \tag{5.26}$$

(One needs to be careful with the notation in the multipartite case, e.g., in (5.26) $\Phi(\hat{F}(\omega, \cdot), \cdot) = \omega$ is actually a state, since $\hat{F}(\omega, \cdot)$ is an effect, etc.) Both faithful state Φ and faithful effect F can be used to express the state–effect pairing, namely

$$\zeta(b) = \Phi(a_\zeta, b) = \hat{F}(\omega_b, \zeta), \quad a_\zeta := \hat{F}(\zeta, \cdot), \ \omega_b := \Phi(b, \cdot), \tag{5.27}$$

or, substituting,

$$\zeta(b) = \Phi(\hat{F}(\zeta, \cdot), b) = \hat{F}(\Phi(b, \cdot), \zeta). \tag{5.28}$$

Equation (5.24) can also be rewritten as follows:

$$F_{23}\Phi_{12} = \alpha\, \mathbf{Swap}_{13}, \tag{5.29}$$

where $\mathbf{Swap}_{ij}$ denotes the transformation swapping S_i with S_j. In Figure 5.4 Postulate FAITHE is illustrated graphically.

Equation (5.29) means that by using the state Φ and the effect F one can achieve probabilistic **teleportation** of states from S_2 to S_4. In fact, one has

$$F_{23}\omega_2\Phi_{34} = F_{23}\Phi_{12}(a_\omega, \cdot)\Phi_{34} = \alpha\Phi_{14}(a_\omega, \cdot) = \alpha\omega_4. \tag{5.30}$$

Using the last identity we can also see that Postulate FAITHE is also equivalent to the identity

$$F_{23}\Phi_{12}\Phi_{34} = \alpha\Phi_{14}, \tag{5.31}$$

which by linearity is extended from local effects to all effects, by virtue of $\mathfrak{E}^{\odot 2} = \mathfrak{E}^{\otimes 2}$. With equivalent notation we can write $(\Phi, \Phi)(\cdot, F, \cdot) = \alpha\Phi$.

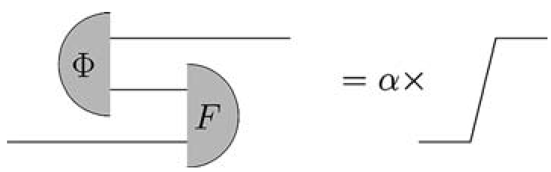


Fig. 5.4 An illustration of Postulate FAITHE.

The effect F is also completely faithful, in the sense that the correspondence $F_{\mathscr{A}} := F \circ (\mathscr{A}', \mathscr{I}) \Longleftrightarrow \mathscr{A}$ is bijective (in finite dimensions). In fact one has

$$[F \circ (\mathscr{A}', \mathscr{I})]_{23}(\Phi, \Phi) = \alpha(\mathscr{A}, \mathscr{I})\Phi, \tag{5.32}$$

and, since Φ is dynamically faithful (it is symmetric preparationally faithful), the correspondence $F_{\mathscr{A}} := F \circ (\mathscr{A}', \mathscr{I}) \Longleftrightarrow \mathscr{A}$ is one-to-one and surjective, whence it is a bijection (in finite dimensions). It is also easy to see that $F \circ (\mathscr{A}', \mathscr{I}) = F \circ (\mathscr{I}, \mathscr{A})$, since

$$\begin{aligned}[F \circ (\mathscr{I}, \mathscr{A})]_{23}(\Phi, \Phi) &= F_{23}(\Phi, (\mathscr{A}, \mathscr{I})\Phi) = F_{23}(\Phi, (\mathscr{I}, \mathscr{A}')\Phi) \\ &= \alpha(\mathscr{I}, \mathscr{A}')\Phi = \alpha(\mathscr{A}, \mathscr{I})\Phi = [F \circ (\mathscr{A}', \mathscr{I})]_{23}(\Phi, \Phi),\end{aligned} \tag{5.33}$$

whence transposition can be equivalently defined with respect to the faithful effect F. The bijection $F_{\mathscr{A}} := F \circ (\mathscr{I}, \mathscr{A}) \Longleftrightarrow \mathscr{A}$ is cone-preserving in both directions, since to every transformation there corresponds an effect, and to each effect $A \in \mathfrak{E}(\mathsf{S}^{\odot 2})$ there corresponds a transformation, since

$$A_{23}(\Phi, \Phi) = \Omega_A = (\mathscr{T}_{\Omega_A}, \mathscr{I})\Phi =: (\mathscr{T}_A, \mathscr{I})\Phi. \tag{5.34}$$

Therefore, the map $\mathscr{A} \mapsto F_{\mathscr{A}}$ realizes the cone-isomorphism $\mathfrak{E}_+(\mathsf{S}^{\odot 2}) \simeq \mathfrak{T}_+(\mathsf{S})$, which is just the composition of the weak self-duality and of the isomorphism $\mathfrak{S}_+(\mathsf{S}^{\odot 2}) \simeq \mathfrak{T}_+(\mathsf{S})$ due to PFAITH. However, as mentioned in footnote 39, the map

$$\mathfrak{E}_+(\mathsf{S}^{\odot 2}) \ni A \mapsto \Omega_A := A_{23}(\Phi, \Phi) \in \mathfrak{S}(\mathsf{S}^{\odot 2}) \tag{5.35}$$

is bijective between $\mathfrak{S}_{\mathbb{F}}(\mathsf{S}^{\odot 2})$ and $\mathfrak{E}_{\mathbb{F}}(\mathsf{S}^{\odot 2})$, but it does not realize the cone-isomorphism $\mathfrak{S}_+(\mathsf{S}^{\odot 2}) \simeq \mathfrak{E}_+(\mathsf{S}^{\odot 2})$, since it is not surjective over $\mathfrak{E}_+(\mathsf{S}^{\odot 2})$. Indeed, for $A \in \mathfrak{E}(\mathsf{S}^{\odot 2})$ physical effect, one has $A_{23}(\Phi, \Phi) = (\mathscr{T}_A, \mathscr{I})\Phi$ with $\mathscr{T}_A \in \mathfrak{T}(\mathsf{S})$ physical transformation. However, there is no guarantee that, vice versa, a physical transformation always has a corresponding physical effect, e.g., for the identity transformation in (5.31). It also follows that any bipartite observable $\mathbb{A} = \{A_l\}$ leads to the **totally depolarizing channel** $\mathscr{T}_{(e,e)}\omega = \chi$, $\forall \omega \in \mathfrak{S}$.[40] Using the faithfulness of F it is possible to achieve probabilistically any transformation on a state ω by performing a joint test on the system interacting with an ancilla, i.e., $(\omega\Phi)(F_{\mathscr{A}'}, \cdot) = \alpha\mathscr{A}\omega$ (for Stinespring-like dilations in an operational context see Ref. [31]).

[40] Indeed, one has $\sum_l (A_l)_{23}\omega_2\Phi_{34} = (e, e)_{23}\omega_2\Phi_{34} = \Phi_{12}(a_\omega, e)\Phi_{34}(e, \cdot) = \omega(e)\chi$.

More about the constant α. Notice that the number $0 < \alpha \leqslant 1$ is the probability of achieving teleportation $\alpha = (F_{23}\omega_2\Phi_{34})(e)$. It is independent of the state ω, and depends only on F, since it is given by $\alpha \equiv \alpha_F = [F_{23}\Phi_{12}\Phi_{34}](e, e)$. The maximum value maximized over all bipartite effects

$$\alpha(\mathsf{S}) = \max_{A \in \mathfrak{E}(\mathsf{S}^{\odot 2})} \{(\Phi, \Phi)(e, A, e)\} \tag{5.36}$$

is a property of the system S only, and depends on the particular probabilistic theory.

More on the relation between Postulates PFAITH and FAITHE. Postulate PFAITH guarantees the existence of a symmetric preparationally faithful state for each pair of identical systems $\mathsf{S}^{\odot 2}$. Now, consider the bipartite system $\mathsf{S}^{\odot 2} \odot \mathsf{S}^{\odot 2}$, and denote by $\mathbf{\Phi}$ a symmetric preparationally faithful state for it. The map $A \mapsto \Omega_A := \mathbf{\Phi}(A, \cdot, \cdot)\ \forall A \in \mathfrak{E}(\mathsf{S}^{\odot 2})$ establishes the state–effect cone-isomorphism for $\mathsf{S}^{\odot 2}$, whence there must exist an effect A_Φ such that

$$\mathbf{\Phi}(A_\Phi, \cdot, \cdot) = \beta\Phi, \quad 0 < \beta \leqslant 1. \tag{5.37}$$

Suppose now that the faithful state can be chosen in such a way that it maps separable states to separable effects as follows:

$$\mathbf{\Phi}(\cdot, \cdot, (a, b)) = \gamma(\omega_a, \omega_b) = \gamma\Phi(\cdot, a)\Phi(\cdot, b), \quad \gamma > 0. \tag{5.38}$$

Then one has

$$\gamma(A_\Phi)_{13}(\Phi, \Phi) = \mathbf{\Phi}(A_\Phi, \cdot, \cdot) = \beta\Phi, \tag{5.39}$$

namely, according to (5.31) one has $\beta^{-1}\gamma A_\Phi \equiv \hat{F}$, which is the effect whose existence is postulated by FAITHE. Notice, however, that the factorization (5.38) doesn't need to be satisfied. In other words, the automorphism relating the cone-isomorphism induced by $\mathbf{\Phi}$ to another cone-isomorphism that preserves local effects may be unphysical (see also footnote 39). One can instead require a stronger version of postulate PFAITH, postulating the existence of a preparationally **super-faithful** symmetric state Φ, also achieving a four-partite preparationally symmetric faithful state $\mathbf{\Phi}$ as $(\Phi, \Phi) = \mathbf{\Phi}$. A weaker version of such a postulate is thoroughly analyzed in Ref. [31], where it is also shown that it leads to Stinespring-like dilations of deterministic transformations.

The case of QM. It is a useful exercise to see how the present framework translates into the quantum case, and find which additional constraints can arise from a specific probabilistic theory. For simplicity we consider a maximally entangled state (with all positive amplitudes in a fixed basis) as a preparationally symmetric state Φ. The corresponding marginal state is given by the density matrix $d^{-1}I$, I

denoting the identity on the Hilbert space. For the constant α one has $\alpha = d^{-2}$, where d is the dimension of the Hilbert space. A simple calculation shows that the identity $\omega_a = \mathscr{T}'_a\chi$ for $\mathscr{T}_a \in a$ translates to[41]

$$\omega_a = \sqrt{\alpha}\varsigma(a), \qquad \Leftarrow \text{in QM}, \tag{5.40}$$

where the involution ς of the Jordan form in (5.20) here is also an automorphism of states/effects, whence identity (5.40) expresses the self-duality of QM. On rewriting (5.40) in terms of the faithful effect F (which would be an element of a Bell measurement), one obtains[42]

$$(\cdot, F)(\Phi, \cdot) = \sqrt{\alpha}|\Phi|, \qquad \Leftarrow \text{in QM}. \tag{5.41}$$

Another feature of QM is that the preparationally faithful symmetric state Φ is super-faithful, namely $\boldsymbol{\Phi} = (\Phi, \Phi)$ is preparationally faithful for $\mathsf{S}^{\odot 4}$.

5.4.2 PURIFY: a postulate on purifiability of all states

In the present section for completeness I briefly explore the consequences of assuming purifiability for all states, namely the following postulate.

Postulate PURIFY (Purifiability of states). *For every state ω of S there exists a pure bipartite state Ω of $\mathsf{S}^{\odot 2}$ having it as marginal state, namely*

$$\forall \omega \in \mathfrak{S}(\mathsf{S}),\ \exists \Omega \in \mathfrak{S}(\mathsf{S}^{\odot 2}) \text{ pure, such that } \Omega(e, \cdot) = \omega. \tag{5.42}$$

Postulate PURIFY has been analyzed in Ref. [31], where the following lemma is proved.

Lemma 6. *If Postulate PFAITH holds, then Postulate PURIFY implies the following assertions.*

(1) *Even without assuming purity of the preparationally faithful state Φ, the identity transformation is atomic, and purity of Φ can be derived.*
(2) $\mathfrak{S}_+ \equiv \mathsf{Erays}(\mathfrak{T}_+)\chi$, *i.e., each state can be obtained by applying an atomic transformation to the marginal state $\chi := \Phi(e, \cdot)$.*
(3) $\mathfrak{E}_+ \equiv e \circ \mathsf{Erays}(\mathfrak{T}_+)$, *i.e., each effect can be achieved with an atomic transformation.*

Points (2) and (3) correspond to the square root of states and effects in the quantum case.

[41] For $\Phi = d^{-1}\sum_{nm} |n\rangle|n\rangle\langle m|\langle m|$ the marginal state is $\chi = d^{-1}I$ and the Jordan involution is the complex conjugation with respect to the orthonormal basis $\{|n\rangle\}$. For quantum operation $\mathscr{T} = \sum_n T_n \cdot T_n^\dagger$ with corresponding effect $a = \sum_n T_n^\dagger T_n$, one has $\mathscr{T}'\chi = d^{-1}\sum_n T_n{}^t T_n^* = d^{-1}\sum_n (T_n^\dagger T_n)^* = \sqrt{\alpha}\varsigma(a)$.

[42] In fact, one has $\omega_a := \Phi(a, \cdot) = \sqrt{\alpha}\varsigma(a)$, namely $\Phi(\varsigma(a), \cdot) = \sqrt{\alpha}a$, i.e., $|\Phi|(a, \cdot) = \sqrt{\alpha}a$, and, using (5.25), one has $\sqrt{\alpha}\hat{F}(\Phi(a, \cdot), \cdot) = |\Phi|(a, \cdot)$, namely the statement.

5.5 What is special about quantum mechanics as a probabilistic theory?

The mathematical representation of the operational probabilistic framework derived up to now is completely general for any fair operational framework that allows local tests, test-calibration, and state preparation. These include not only QM and classical–quantum hybrid, but also other non-signaling non-local probabilistic theories such as the *PR-boxes* theories [20]. Postulate PFAITH has proved to be remarkably powerful, implying (1) the local observability principle, (2) the tensor-product structure for the linear spaces of states and effects, (3) weak self-duality, (4) realization of all states as transformations of the marginal faithful state $\Phi(e, \cdot)$, (5) locally indistinguishable ensembles of states corresponding to local observables – i.e., EPR-cheating in bit commitment – and more. By adding FAITHE one even has teleportation! However, despite all these positive landmarks, it is still unclear whether one can derive QM from these principles only.

What is then special about QM? The peculiarity of QM among probabilistic operational theories is the following.

Effects can not only be linearly combined, but also can be composed of each other, so that complex effects make a C^*-algebra.

Operationally the last assertion is odd, since *the notion of effect abhors composition!* Therefore, the composition of effects (i.e., the fact that they make a C^*-algebra, i.e., an operator algebra over complex Hilbert spaces) must be derived from additional postulates. What I will show here is the following.

With a single mathematical postulate, and assuming atomicity of evolution, one can derive the composition of effects in terms of composition of atomic events.

One thus is left with the problem of translating the remaining mathematical postulate into an operational one. Let's now examine the two postulates.

Postulate AE (Atomicity of evolution). *The composition of atomic transformations is atomic.*

This postulate is so natural that it looks obvious.[43] However, even though for atomic events $\mathscr{A}$ and $\mathscr{B}$ the event $\mathscr{C} = \mathscr{B} \circ \mathscr{A}$ is not refinable in the corresponding cascade-test, there is no guarantee that $\mathscr{C}$ is not refinable in any other test. We remember that mathematically atomic events belong to $\mathsf{Erays}(\mathfrak{T}_+)$, the extremal rays of the cone of transformations.

We now state the mathematical postulate.

[43] Indeed, when joining events $\mathscr{A}$ and $\mathscr{B}$ into the event $\mathscr{A} \wedge \mathscr{B}$, the latter is atomic if both $\mathscr{A}$ and $\mathscr{B}$ are atomic.

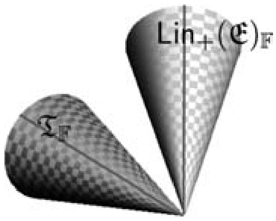


Fig. 5.5 The Choi–Jamiolkowski isomorphism between the cone $\mathfrak{T}_+$ of physical transformations and the cone $\mathsf{Lin}_+(\mathfrak{E}_{\mathbb{C}})$ of positive matrices over complex effects establishes a one-to-one correspondence between extremal-ray points of the two cones, identifying effects (modulo a phase) with atomic transformations (the lines over the cones represent a pair of corresponding rays).

Mathematical Postulate CJ (Choi–Jamiolkowski isomorphism (Figure 5.5)). *The cone of transformations is isomorphic*[44] *to the cone of positive bilinear forms over complex effects [27, 28], i.e.,* $\mathfrak{T}_+ \simeq \mathsf{Lin}_+(\mathfrak{E}_{\mathbb{C}})$.

In terms of a sesquilinear scalar product over complex effects, positive bilinear forms can be regarded as positive matrices over complex effects, i.e., elements of the cone $\mathsf{Lin}_+(\mathfrak{E}_{\mathbb{C}})$.

The extremal rays $\mathsf{Erays}(\mathsf{Lin}_+(\mathfrak{E}_{\mathbb{C}}))$ are rank-one positive operators $|x\rangle\langle x| \in \mathsf{Erays}(\mathsf{Lin}_+(\mathfrak{E}_{\mathbb{C}}))$ with $x \in \mathfrak{E}_{\mathbb{C}}$, and the map $\pi : x \mapsto \pi(x) := |x\rangle\langle x|$ is surjective over $\mathsf{Erays}(\mathsf{Lin}_+(\mathfrak{E}_{\mathbb{C}}))$. One has $\pi(x e^{i\phi}) = \pi(x)$, and $\pi^{-1}(|x\rangle\langle x|) = \{e^{i\phi}x\} \subseteq \mathfrak{E}_{\mathbb{C}}$, i.e., the set of complex effects mapped to the same rank-one positive operator is the set of complex effects that differ only by a multiplicative phase factor. We will denote by $|x| \in \mathfrak{E}_{\mathbb{C}}$ a fixed choice of representative for such an equivalence class,[45] introduce the phase corresponding to such a choice as $x =: |x|e^{i\phi(x)}$, and denote by $\mathfrak{E}_{\mathbb{C}}/\phi$ the set of equivalence classes, or, equivalently, of their representatives. Now, since the representatives $|x| \in \mathfrak{E}_{\mathbb{C}}/\phi$ are in one-to-one correspondence with the points on $\mathsf{Erays}(\mathsf{Lin}_+(\mathfrak{E}_{\mathbb{C}}))$, the CJ isomorphism establishes a bijective map between $\mathfrak{E}_{\mathbb{C}}/\phi$ and $\mathsf{Erays}(\mathfrak{T}_+)$ as follows:

$$\tau\colon \mathfrak{E}_{\mathbb{C}}/\phi \ni |x| \leftrightarrow \tau(|x|) \in \mathsf{Erays}(\mathfrak{T}_+). \tag{5.43}$$

5.5.1 Building up an associative algebra structure for complex effects

Assuming Postulate AE, we can introduce an associative composition between the effects in $\mathfrak{E}_{\mathbb{C}}/\phi$ via the bijection τ,

$$|a||b| := \tau^{-1}(\tau(|a|) \circ \tau(|b|)). \tag{5.44}$$

[44] For the definition of cone-isomorphisms, see footnote 34.
[45] An example of choice of representative is given by $||x|\rangle := \langle e_{\iota(x)}|\pi(x)|e_{\iota(x)}\rangle^{-1/2}\pi(x)|e_{\iota(x)}\rangle$, namely $|x| := |(x, e_{\iota(x)})|^{-1}(x, e_{\iota(x)})x$, with $\iota(x) = \min\{i\colon (x, e_i) \neq 0\}$, for given fixed basis for $\mathfrak{E}_{\mathbb{C}}$.

Notice that, by definition, $|a||b|$ is a representative of an equivalence class in $\mathfrak{E}_{\mathbb{C}}$, whence $|(|a||b|)| = |a||b|$. The above composition is extended to all elements of $\mathfrak{E}_{\mathbb{C}}$ by taking

$$ab := |a||b|\mathrm{e}^{\mathrm{i}\phi(a)}\mathrm{e}^{\mathrm{i}\phi(b)}, \tag{5.45}$$

and, since $|(|a||b|)| = |a||b|$, one has $|ab| = |a||b|$, and $\phi(ab) = \phi(a) + \phi(b)$. It follows that the extension is itself associative, since

$$\begin{aligned}(ab)c &= |ab||c|\mathrm{e}^{\mathrm{i}\phi(ab)+\mathrm{i}\phi(c)} = |a||b||c|\mathrm{e}^{\mathrm{i}\phi(a)+\mathrm{i}\phi(b)+\mathrm{i}\phi(c)} \\ &= |a||bc|\mathrm{e}^{\mathrm{i}\phi(a)+\mathrm{i}\phi(bc)} = a(bc).\end{aligned} \tag{5.46}$$

The composition is also distributive with respect to the sum, since it follows the same rules as those of complex numbers. We will denote by ι the identity in $\mathfrak{E}_{\mathbb{C}}/\phi$ when it exists, which also works as an identity for multiplication of effects as in (5.45). Notice that, since the identity transformation $\mathscr{I}$ is atomic, one has $\iota := \tau^{-1}(\mathscr{I}) \in \mathfrak{E}_{\mathbb{C}}/\phi$ according to (5.44).

5.5.2 Building up a C^*-algebra structure over complex effects

We want now to introduce a notion of adjoint for effects. We will do this in two steps: (a) we introduce an antilinear involution on the linear space $\mathfrak{E}_{\mathbb{C}}$; (b) we extend the associative product (5.45) under such antilinear involution.

(a) First we notice that the complex space $\mathfrak{E}_{\mathbb{C}}$ has been constructed as $\mathfrak{E}_{\mathbb{C}} = \mathfrak{E}_{\mathbb{R}} \oplus i\mathfrak{E}_{\mathbb{R}}$ starting from real combinations of physical effects $\mathfrak{E}_{\mathbb{R}} = \mathsf{Span}_{\mathbb{R}}(\mathfrak{E}_+)$, i.e., one has the unique Cartesian decomposition $x = x_{\mathrm{R}} + \mathrm{i}x_{\mathrm{I}}$ of $x \in \mathfrak{E}_{\mathbb{C}}$ in terms of $x_{\mathrm{R}}, x_{\mathrm{I}} \in \mathfrak{E}_{\mathbb{R}}$. We can then define the antilinear *dagger* involution $\dagger$ on $\mathfrak{E}_{\mathbb{C}}$ by taking $x^\dagger = x\ \forall x \in \mathfrak{E}_{\mathbb{R}}$ and $x^\dagger := x_{\mathrm{R}} - \mathrm{i}x_{\mathrm{I}}\ \forall x \in \mathfrak{E}_{\mathbb{C}}$. Notice that $\mathfrak{E}_{\mathbb{C}}$ is closed under such involution. On taking the involution of the defining identity $x =: |x|\mathrm{e}^{\mathrm{i}\phi(x)}$ one has $|x^\dagger| = |x|^\dagger\mathrm{e}^{-\mathrm{i}\phi(x^\dagger)-\mathrm{i}\phi(x)}$, which is consistently satisfied by choosing $|x^\dagger| = |x|^\dagger$ and $\phi(x^\dagger) = -\phi(x)\ \forall x \in \mathfrak{E}_{\mathbb{C}}$ (these identities are satisfied, e.g., for the choice of representative in footnote 45).

(b) The multiplications $a^\dagger b$ and $ab^\dagger$ are defined via the scalar product over $\mathfrak{E}_{\mathbb{C}}$ as follows:[46]

$$\forall c \in \mathfrak{E}_{\mathbb{C}}: \quad (c, a^\dagger b) := (ac, b), \qquad (c, ab^\dagger) := (cb, a). \tag{5.47}$$

This is possible since the scalar product over $\mathfrak{E}_{\mathbb{C}}$ is supposed to be non-degenerate. It is then easy to verify that one has the identities $(ab)^\dagger = b^\dagger a^\dagger$ and $\iota^\dagger = \iota$.

In this way $\mathfrak{E}_{\mathbb{C}}$ is closed under complex linear combinations, the adjoint, and associative composition, and possibly contains the identity element ι; that is, it is an

[46] The right and left multiplications are just special elements of the algebra $\mathsf{Lin}(\mathfrak{E}_{\mathbb{C}})$, whence their adjoints are definable via the scalar product as usual.

associative complex algebra with adjoint, closed with respect to the adjoint. The scalar product on $\mathfrak{E}_{\mathbb{C}}$ in conjunction with the identity leads to a strictly positive linear form over $\mathfrak{E}_{\mathbb{C}}$, defined as $\Phi = (\iota, \cdot)$, and one has $\Phi(a^\dagger b) = (\iota, a^\dagger b) = (a, b)$.[47] Such a form is also a *trace*, i.e., it satisfies the identity $\Phi(ba) = \Phi(ab)$, which can be easily verified using definitions (5.47).[48] The complex linear space of the algebra closed with respect to the norm induced by the scalar product makes it a Hilbert space, and the action of the algebra over itself regarded as a Hilbert space makes it an operator algebra.[49] It is a standard result of the theory of operator algebras that the closure of $\mathfrak{E}_{\mathbb{C}}$ under the operator norm (which is guaranteed in finite dimensions) is a C^*-algebra. We have therefore built a C^*-algebra structure over the complex linear space of effects $\mathfrak{E}_{\mathbb{C}}$. This is the *cyclic representation* [39] given by

$$\Phi(a) = \langle \iota | \pi_\Phi(a) | \iota \rangle, \tag{5.48}$$

π_Φ denoting the algebra representation corresponding to Φ.[50] In our case one has $\pi_\Phi(a)|\iota\rangle = |a\rangle$, along with the *trace* property $\langle \iota | \pi_\Phi(a)\pi_\Phi(b) | \iota \rangle = \langle \iota | \pi_\Phi(b)\pi_\Phi(a) | \iota \rangle$. The latter can be actually realized as a trace as $\Phi(a^\dagger b) = \mathrm{Tr}\,[O(a)^\dagger O(b)]$, via a faithful representation O: $a \mapsto O(a) \in \mathsf{Lin}(\mathsf{H})$ of the algebra $\mathfrak{E}_{\mathbb{C}}$ as a sub-algebra of $\mathsf{Lin}(\mathsf{H})$ of operators over a Hilbert space H with dimension $\dim(\mathsf{H})^2 \geqslant \dim(\mathfrak{E}_{\mathbb{C}})$. In this way, one has $\pi_\Phi(a) = (O(a) \otimes I)$ with the cyclic vector represented as $|\iota\rangle = \sum_n |n\rangle \otimes |n\rangle$, $\{|n\rangle\}$ being any orthonormal basis for H.

5.5.3 Recovering the action of transformations over effects

In order to complete the mathematical representation of the probabilistic theory, we now need to define the action of the elements of $\mathfrak{T}_{\mathbb{C}}$ over $\mathfrak{E}_{\mathbb{C}}$, and to select the cone of physical transformations $\mathfrak{T}_+$. We will show that $\mathfrak{T}_+$ is given by the completely positive linear maps on $\mathfrak{E}_{\mathbb{C}}$, namely the linear maps of the Kraus form, i.e., the atomic transformations act on $x \in \mathfrak{E}_{\mathbb{C}}$ as $x \circ \tau(|a|) = |a|^\dagger x |a| \equiv a^\dagger x a$.

First, notice that the full span $\mathsf{Lin}(\mathfrak{E}_{\mathbb{C}})$ is recovered from $\mathsf{Erays}(\mathsf{Lin}_+(\mathfrak{E}_{\mathbb{C}}))$ via the polarization identity

$$|a\rangle\langle b| = \frac{1}{4}\sum_{k=0}^{3} \mathrm{i}^k |(a + \mathrm{i}^k b)\rangle\langle(a + \mathrm{i}^k b)|. \tag{5.49}$$

[47] The form is strictly positive since $\Phi(a^\dagger a) = (a, a) \geqslant 0$, with the equals sign only if $a = 0$, since the scalar product is non-degenerate.
[48] One has $\Phi(ab) = (\iota, ab) = (\iota, a(b^\dagger)^\dagger) = (b^\dagger, a)$ and $\Phi(ba) = (\iota, ba) = (\iota, (b^\dagger)^\dagger a) = (b^\dagger, a)$.
[49] This construction is a special case of the Gelfand–Naimark–Segal (GNS) construction [40], in which the form Φ is a trace. In the standard GNS construction the form Φ may be degenerate, i.e., one can have $\Phi(a^\dagger a) = 0$ for some $a \neq 0$, and the vectors of the representation are built up as equivalence classes modulo vectors having $\Phi(a^\dagger a) = 0$.
[50] This means that $\pi_\Phi(a)\pi_\Phi(b) = \pi_\Phi(ab)$ and $\pi_\Phi(a^\dagger) = \pi_\Phi(a)^\dagger$.

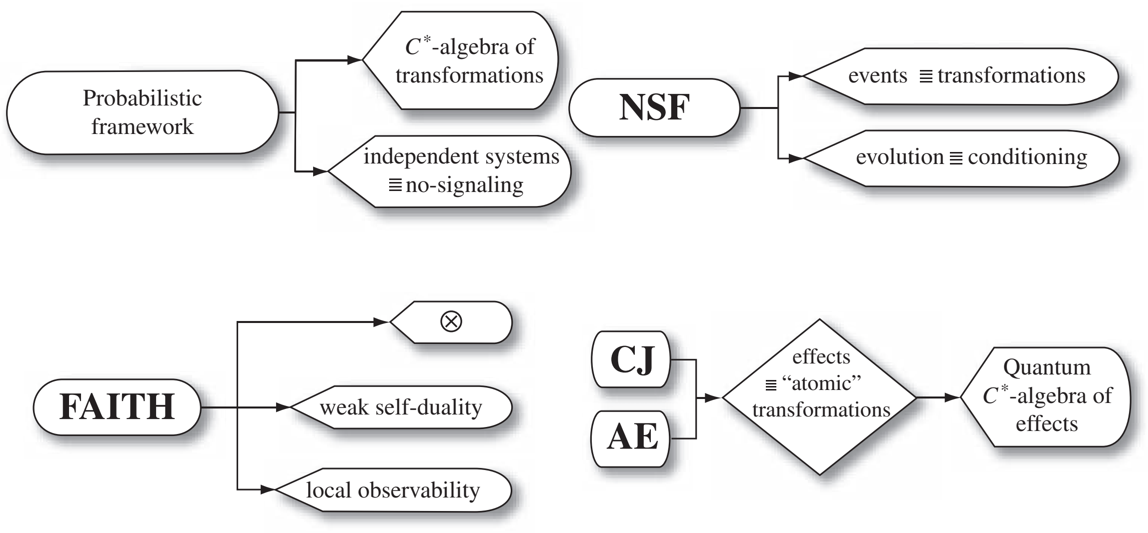


Fig. 5.6 An operational axiomatic framework for quantum mechanics: a summary of the relevant logical implications.

Correspondingly, we introduce the generalized transformations

$$\tau(b,a) := \frac{1}{4}\sum_{k=0}^{3} \mathrm{i}^k \tau(|a+\mathrm{i}^k b|) \in \mathfrak{T}_{\mathbb{C}}. \tag{5.50}$$

The map

$$|a\rangle\langle b| \mapsto \chi(|a\rangle\langle b|) := b^\dagger \cdot a \tag{5.51}$$

is a CJ isomorphism: it represents a bijective map between the cones $\mathsf{Lin}_+(\mathfrak{E}_{\mathbb{C}})$ and $\mathfrak{T}_+$, which can be extended to a cone-preserving linear bijection between $\mathsf{Lin}(\mathfrak{E}_{\mathbb{C}})$ and $\mathfrak{T}_{\mathbb{C}} \equiv \mathsf{Lin}(\mathfrak{E}_{\mathbb{C}})$.[51] As a consequence of (5.44), the CJ isomorphism $\tau : |a| \mapsto \tau(|a|)$ will differ from the isomorphism χ by an automorphism $\mathcal{U}$ of the C^*-algebra of effects; that is, one has $x \circ \tau(|a|) = \mathcal{U}(a^\dagger)x\mathcal{U}(a)$, with $\mathcal{U}(a) = u^\dagger a u$ with $uu^\dagger = u^\dagger u = \iota$. It follows that the probabilistic equivalence classes are given by $[\tau(|a|)]_{\mathrm{eff}} = e \circ \tau(|a|) = u^\dagger a^\dagger a u$. Notice that $[\tau(\iota)]_{\mathrm{eff}} = u^\dagger \iota^\dagger \iota u = \iota$; that is, ι coincides with the deterministic effect $\iota = e$. Complex effects are thus recovered from atomic transformations via the identity $e \circ \tau(e,a) = u^\dagger a u$. Figure 5.6 is a flow diagram summarizing the relevant logical implications of the present operational axiomatic framework for QM.

[51] This can be directly checked using the operator algebra representation built over $\mathfrak{E}_{\mathbb{C}}$, whereas the isomorphism corresponds to the map $O(b^\dagger x a) = \chi(|a\rangle\langle b|)(x) = \mathrm{Tr}_1[(O(x)\otimes I)|a\rangle\langle b|]$, and, reversely, $|a\rangle\langle b| = \chi^{-1}(\tau(b,a)) = (\tau(b,a)\otimes \mathscr{I})(|\iota\rangle\langle\iota|)$.

5.5.4 Reconstructing quantum mechanics from the probabilistic theory

It is now possible to reconstruct from the probability tables of the systems the full C^*-algebra of complex effects $\mathfrak{E}_{\mathbb{C}}$ as an operator algebra $\mathfrak{E}_{\mathbb{C}} \subseteq \oplus_i \mathsf{Lin}(\mathsf{H_i})$. Here is the recipe.

(1) Look for all sub-cones $(\mathfrak{E}_+)_i$ invariant under $\mathfrak{T}_+$.
Then, for each i:
(2) introduce a complex Hilbert space H_i such that $(\mathfrak{E}_{\mathbb{C}})_i \subseteq \mathsf{Lin}(\mathsf{H_i})$, i.e., with $\dim(\mathsf{H}_i) = \left\lceil \sqrt{\dim[(\mathfrak{E}_{\mathbb{C}})_i]} \right\rceil$, $\lceil x \rceil$ the smallest integer greater than x;
(3) represent e as the identity over $\oplus_i \mathsf{H}_i$;
(4) build $(\mathfrak{T}_{\mathbb{C}})_i \subseteq \mathsf{Lin}(\mathsf{Lin}(\mathsf{H_i}))$;
(5) look for atomic transformations $\mathsf{Erays}(\mathfrak{T}_+)_i$;
(6) for a given atomic transformation $\mathscr{A} \in \mathsf{Erays}(\mathfrak{T}_+)_i$ take an operator $A \in \mathsf{Lin}(\mathsf{H_i})$ to represent $\mathscr{A}$ as $A^\dagger \cdot A \in \mathsf{Lin}(\mathsf{Lin}(\mathsf{H_i}))$;
(7) represent $[\mathscr{A}]_{\text{eff}}$ as $A^\dagger A$;
(8) repeat steps 6 and 7 for another transformation $\mathscr{B}$;
(9) compose $\mathscr{C} = \mathscr{B} \circ \mathscr{A}$ and represent $\mathscr{C}$ as $C^\dagger \cdot C$, with $C = AB$;
(10) repeat steps 8 and 9 to build the whole algebra of effects and the corresponding representation of the algebra of transformations; and
(11) construct states as density operators using the Gleason-like theorem [41] for effects [42, 43].

5.6 Conclusions

Theoretical physics should be, in essence, a mathematical "representation" of reality. By "representation" we mean describing one thing by means of another, to connect the object that we want to understand – the *thing-in-itself* – with an object that we already know well – the *standard*. In theoretical physics we lay down morphisms from structures of reality to corresponding mathematical structures: groups, algebras, vector spaces, etc., each mathematical structure capturing a different side of reality.

Quantum mechanics somehow goes differently. We have a beautiful simple mathematical structure – Hilbert spaces and operator algebras – with unprecedented predictive power in the entire physical domain. However, we don't have morphisms from the operational structure of reality into a mathematical structure. In this sense we can say that QM is not yet truly a "representation" of reality. A large part of the formal structure of QM is a set of formal tools for describing the process of gathering information in any experiment, independently of the particular physics involved. It is mainly a kind of information theory, a theory about our knowledge of physical entities rather than about the entities themselves. If we were to strip off this informational part from the theory, what

Table 5.1 *A summary of notation*

Symbol(s)	Meaning	Related quantities
$\mathsf{S}_1 \odot \mathsf{S}_2$	Bipartite system obtained by composing S_1 with S_2	
$\mathsf{S} = \{\mathbb{A}, \mathbb{B}, \mathbb{C}, \ldots\}$	System	
$\mathbb{A}, \mathbb{B}, \mathbb{C}, \ldots$	Tests	$\mathbb{A} = \{\mathscr{A}_j\}$, Test := set of possible events
$\mathscr{A}, \mathscr{B}, \mathscr{C}, \ldots$	Events $\equiv$ transformations	
ω	States, $\mathfrak{S}$ convex set of states	$\omega(\mathscr{A})$: probability that event $\mathscr{A}$ occurs in state ω
$\mathfrak{T}$	Convex monoid of transformations/events	$\mathfrak{T}_{\mathbb{R}}$,$\mathfrak{T}_{\mathbb{C}}$: linear spans of $\mathfrak{T}$, $\mathfrak{T}_+$: convex cone
$[\mathscr{A}]_{\text{eff}}$	Effect containing event $\mathscr{A}$	
$a, b, c, \ldots$	Effects	e: deterministic effect
$\mathfrak{E}$	Convex set of effects	$\mathfrak{E}_{\mathbb{R}}$, $\mathfrak{E}_{\mathbb{C}}$: linear spans of $\mathfrak{E}$, $\mathfrak{E}_+$: convex cone
$\mathbb{L} = \{l_j\}$	observable	$\sum_{l_i \in \mathbb{L}} l_i = e$
$\mathfrak{T}_{\mathbb{C}}$	C^*-algebra of transformations/events	
$a \circ \mathscr{T}$	Operation of transformation $\mathscr{T}$ over effect a	
$\omega_{\mathscr{A}}$	Conditioned states	$\omega_{\mathscr{A}} := \omega(\cdot \circ \mathscr{A})/\omega(\mathscr{A})$, $\mathscr{A}\omega = \omega(\cdot \circ \mathscr{A})$
$\mathsf{Lin}_+(\mathfrak{E}_{\mathbb{C}})$	Cone of linear maps corresponding to positive bilinear forms over $\mathfrak{E}_{\mathbb{C}}$	$\mathsf{Lin}_+(\mathfrak{E}_{\mathbb{C}}) = \{\mathscr{T} \in \mathfrak{T}_{\mathbb{C}} : (a, a \circ \mathscr{T}) \geqslant 0, \forall a \in \mathfrak{E}_{\mathbb{C}}\}$

would be left should be the true general principle from which QM should be derived.

In the present work I have analyzed the possibility of deriving QM as the mathematical representation of a fair operational framework made of a set of rules that allows one to make predictions about future events on the basis of suitable tests. The two postulates NSF and PFAITH need to be satisfied by an operational framework that is fair, the former in order for one to be able to make predictions that are based on present tests, the latter to allow calibrability of any test and preparability of any state. We have seen that all theories satisfying NSF admit a C^*-algebra representation of events as linear transformations of complex effects. On the basis of a very general notion of dynamical independence, all such theories are *non-signaling*. The C^*-algebra representation of events is just the informational part of the theory. We have then added Postulate PFAITH. Postulate PFAITH has been proved to be remarkably powerful, implying the local observability principle, the tensor-product structure for the linear spaces of states and effects, weak self-duality, and a list of features such as realization of all states as transformations of the marginal faithful state $\Phi(e, \cdot)$, locally indistinguishable ensembles of states corresponding to local observables – i.e., EPR-cheating in bit commitment, and more. We have then explored a postulate dual to PFAITH, Postulate FAITHE for effects, thus deriving additional quantum features, such as teleportation. We feel that we are really close to QM: maybe we are already there and we only need to prove it! All the consequences of these postulates need to be explored further. I have also reported some consequences of a postulate about the purifiability of all states. In any case, we have seen that, whatever the missing postulate is, it must establish a one-to-one correspondence between complex effects and atomic transformations, which, assuming atomicity of evolution (Postulate AE) will make also effects a C^*-algebra. This is what is special about QM (and all hybrid quantum–classical theories), and will exclude other non-signaling probabilistic theories of the kind of the PR boxes.[52] We have seen that the correspondence between effects and atomic transformations is established by the Choi–Jamiolkowski isomorphism, which is hoped to be not too far from an operational principle.

The notation used is summarized in Table 5.1

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[52] The PR boxes in principle satisfy NSF, and can admit a dynamical faithful state, e.g., the boxes of Ref. [44] (private discussion with Tony Short).

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