Realization schemes for quantum instruments in finite dimensions

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We present a general dilation scheme for quantum instruments with continuous outcome space in finite dimensions, in terms of a measurement on a finite-dimensional ancilla, described by a positive operator valued measure (POVM). The general result is then applied to a large class of instruments generated by operator frames, which contains group-covariant instruments as a particular case and allows one to construct dilation schemes based on a measurement on the ancilla followed by a conditional feed-forward operation on the output. In the case of tight operator frames, our construction generalizes quantum teleportation and telecloning, producing a whole family of generalized teleportation schemes in which the instrument is realized via a joint POVM at the sender combined with a conditional feed-forward operation at the receiver. © *2009 American Institute of Physics.*
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I. INTRODUCTION

The theory of quantum measurements for discrete spectrum has been formulated by von Neumann in the pioneering work.[1] For continuous outcome quantum measurements, however, a satisfactory theory has been laking for another 30 years, when the problem was finally settled by Ozawa.[2] The main difficulties with the continuous-outcome measurements were (i) the issue of repeatability and (ii) the compatibility between the statistics of the measurement and the dynamical evolution of the observed system and the measuring apparatus. In their pioneering work,[3] Davies and Lewis introduced an operational framework for the statistical description based on the mathematical concept of "instrument"—i.e., of transformation-valued measure. In this framework they formulated a weak repeatability hypothesis and conjectured that instruments for continuous-outcome measurements can never be repeatable, even in such weak sense. Davies and Lewis, however, overlooked the requirement of complete positivity of the state change in measurements, which was instead established by Kraus[4] in the particular case of yes-no measurements. Thirteen years later Ozawa[2] showed that the state change due to an arbitrary measuring process is described by completely positive (CP) instruments, and, vice versa, that any CP instrument can be dilated to an indirect measurement process, with the measured system unitarily interacting with an ancilla which then undergoes the measurement of a von Neumann observable with the same outcome space of the instrument. In the same paper, Ozawa finally proved the Davies and Lewis conjecture for CP instruments, showing that they cannot be weakly repeatable unless their outcome space is discrete.

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A von Neumann observable with continuous-outcome space is a projection-valued measure (PVM), such as the spectral measure of a self-adjoint operator with continuous spectrum. Such a continuous outcome observable can exist only for infinite-dimensional systems. It follows that the Ozawa dilations of quantum instruments with continuous-outcome space, even for finite-dimensional systems, require an infinite-dimensional ancilla. A general positive operator valued measure (POVM), on the contrary, can have a continuous-outcome space even for finite dimensions, e.g., for the measurement of the spin direction.[5] Recently, in Refs. 6 and 7, it has been shown that in finite dimensions every continuous-outcome POVM can be achieved as the randomization of finite-outcome POVMs with no more than d^2 outcomes, d being the dimension of the system's Hilbert space. Exploiting Naimark dilation[8] of the finite-outcome POVMs involved in the randomization, this implies that for finite dimensions any continuous-outcome POVM can be realized as a randomized observable with dimensions no greater than d^2. Therefore, realization of an instrument via indirect measurement of a POVM on a finite-dimensional ancilla allows one to achieve the instrument as the indirect measurement of a randomized observable in finite dimension.

The existence of the dilation for the continuous-outcome instrument to a POVM on a finite-dimensional ancilla has not been considered yet in the literature, and it is not *a priori* obvious, since the usual dilation procedure exploits the orthogonality of the PVM. Such an indirect POVM realization of the instrument is the main result of the present paper, where we construct a general realization scheme for a quantum instrument with continuous outcome space in finite dimension, in terms of an indirect POVM measurement performed on an ancilla interacting with the system. In addition, in this paper we define the notion of instruments generated by operator frames and specialize our dilation theorem to this case, showing that any such instrument allows a realization in terms of an ancilla measurement followed by a conditional feed-forward operation. For tight operator frames, the feed-forward scheme becomes a generalized teleportation scheme, namely, a scheme where a sender performs a joint POVM measurement on the input system and locally on another system of an entangled pair and communicates the measurement outcome to the receiver, who then performs a suitable conditional quantum channel on the other system of the entangled pair. The notions of instruments and channels generated by frames and the related feed-forward realization schemes provide a general framework encompassing a great deal of existing experimental schemes[9–12] and theoretical proposals, such as telecloning[13] and universal-NOT.[14]

The paper is organized as follows. in Sec. II we recall the preliminary notions used in the paper, also giving a new compact rule for expressing the minimal Stinespring dilation of a given CP map as a function of its Choi–Jamiołkowski (CJ) operator. The general dilation theorem for quantum instruments in finite dimensions is then presented in Sec. III, where we construct an indirect measuring process based on a POVM on a finite-dimensional ancilla. In Sec. IV we introduce a class of instruments generated by operator frames, which contains group-covariant instruments as a special case. The case of group-covariant instruments is then analyzed in detail in Sec. V. In Sec. VI we specialize to the case of instruments generated by tight operator frames, showing that any such instrument can be realized via generalized teleportation scheme, with a joint measurement at the sender and a conditional operation at the receiver. This construction generalizes quantum teleportation and provides the general framework for quantum tasks such as telecloning[13] and tele-UNOT,[14] as shown in Sec. VII.

II. GENERAL NOTIONS ON QUANTUM INSTRUMENTS AND CHANNELS

In the following we will denote by $\mathsf{Lin}(\mathsf{H})$ the vector space of linear operators on the Hilbert space H and by $\mathsf{Lin}(\mathsf{H},\mathsf{K})$ the vector space of linear operators from H to K. We will exclusively consider finite-dimensional Hilbert spaces. Moreover, we will denote by $\mathsf{S}(\mathsf{H})$ the convex set of density matrices on H and by $\mathsf{CP}(\mathsf{H},\mathsf{K})$ the convex cone of CP maps from $\mathsf{Lin}(\mathsf{H})$ to $\mathsf{Lin}(\mathsf{K})$.

A. Quantum operations

In quantum mechanics the most general evolution of a system is described by a quantum operation.[15] We will consider generally different input and output systems in the evolution and denote by H_{in} and $\mathsf{H}_{\mathrm{out}}$ the corresponding Hilbert spaces. Then, a quantum operation with input space H_{in} and output space $\mathsf{H}_{\mathrm{out}}$ is a CP map $\mathcal{E} \in \mathsf{CP}(\mathsf{H}_{\mathrm{in}}, \mathsf{H}_{\mathrm{out}})$ which is also trace nonincreasing. The operation $\mathcal{E}$ transforms the input state $\rho_{\mathrm{in}} \in \mathsf{S}(\mathsf{H}_{\mathrm{in}})$ into the output state $\rho_{\mathrm{out}} \in \mathsf{S}(\mathsf{H}_{\mathrm{out}})$ as follows:

$$\rho_{\mathrm{out}} = \frac{\mathcal{E}(\rho_{\mathrm{in}})}{\mathrm{Tr}[\mathcal{E}(\rho_{\mathrm{in}})]}, \tag{1}$$

the transformation occurring with probability $p_{\mathcal{E}} := \mathrm{Tr}[\mathcal{E}(\rho_{\mathrm{in}})]$ among a set of possible transformations. In the deterministic case the map $\mathcal{E}$ is trace preserving, and the quantum operation is usually called quantum channel.

B. Representations of operators and CP maps

1. Operators and bipartite vectors

In finite dimensions it is convenient to exploit the isomorphism between $\mathsf{Lin}(\mathsf{H}_{\mathrm{in}}, \mathsf{H}_{\mathrm{out}})$ and $\mathsf{H}_{\mathrm{out}} \otimes \mathsf{H}_{\mathrm{in}}$ induced by the linear map

$$F \in \mathsf{Lin}(\mathsf{H}_{\mathrm{in}}, \mathsf{H}_{\mathrm{out}}) \mapsto |F\rangle\rangle := (F \otimes \mathbb{1}_{\mathrm{in}})|\mathbb{1}_{\mathrm{in}}\rangle\rangle \in \mathsf{H}_{\mathrm{out}} \otimes \mathsf{H}_{\mathrm{in}}, \tag{2}$$

where $|\mathbb{1}_{\mathrm{in}}\rangle\rangle \in \mathsf{H}_{\mathrm{in}}^{\otimes 2}$ is the maximally entangled vector $|\mathbb{1}_{\mathrm{in}}\rangle\rangle := \Sigma_n |e_n\rangle|e_n\rangle$ defined by the choice of a distinguished orthonormal basis $\{|e_n\rangle\}_{n=1}^{d_{\mathrm{in}}}$ for any copy of H_{in}.

Fixing an orthonormal basis $\{|c_m\rangle\}_{m=1}^{d_{\mathrm{out}}}$ for any copy of $\mathsf{H}_{\mathrm{out}}$, the transpose and the complex conjugate of F are uniquely defined through the relations

$$(\mathbb{1}_{\mathrm{out}} \otimes F^{\tau})|\mathbb{1}_{\mathrm{out}}\rangle\rangle = |F\rangle\rangle, \tag{3}$$

$$F^{*} = (F^{\dagger})^{\tau}, \tag{4}$$

where $|\mathbb{1}_{\mathrm{out}}\rangle\rangle = \Sigma_m |c_m\rangle|c_m\rangle$ and $F^{\dagger} \in \mathsf{Lin}(\mathsf{H}_{\mathrm{out}}, \mathsf{H}_{\mathrm{in}})$ is the adjoint of F. Definitions (2) and (3) imply the elementary identities

$$(B \otimes A)|F\rangle\rangle = |BFA^{\tau}\rangle\rangle,$$

$$\langle\langle F|G\rangle\rangle = \mathrm{Tr}[F^{\dagger}G] := \langle F, G\rangle_{\mathrm{HS}}, \tag{5}$$

where B and A are arbitrary operators in $\mathsf{Lin}(\mathsf{H}_{\mathrm{out}})$ and $\mathsf{Lin}(\mathsf{H}_{\mathrm{in}})$, respectively, and $\langle \cdot, \cdot \rangle_{\mathrm{HS}}$ denotes the Hilbert–Schmidt scalar product in $\mathsf{Lin}(\mathsf{H}_{\mathrm{in}}, \mathsf{H}_{\mathrm{out}})$.

2. Linear maps and bipartite operators

In finite dimensions it is convenient to represent linear maps $\mathcal{M}$ from $\mathsf{Lin}(\mathsf{H}_{\mathrm{in}})$ to $\mathsf{Lin}(\mathsf{H}_{\mathrm{out}})$ as linear operators $R_{\mathcal{M}}$ on $\mathsf{H}_{\mathrm{out}} \otimes \mathsf{H}_{\mathrm{in}}$ via the so-called Choi-Jamiołkowski (CJ) isomorphism,[16,17]

$$R_{\mathcal{M}} = (\mathcal{M} \otimes \mathcal{I})(|\mathbb{1}_{\mathrm{in}}\rangle\rangle\langle\langle \mathbb{1}_{\mathrm{in}}|), \quad \mathcal{M}(\rho) = \mathrm{Tr}_{\mathrm{in}}[(\mathbb{1}_{\mathrm{out}} \otimes \rho^{\tau})R_{\mathcal{M}}], \tag{6}$$

where $\mathcal{I}$ is the identity map and $\mathrm{Tr}_{\mathrm{in}}$ denotes the partial trace on H_{in} [see Ref. 18 for the second equality in Eq. (6)]. The transpose and the complex conjugate of a map are uniquely defined by through the relations

$$(\mathcal{I} \otimes \mathcal{M}^{\tau})(|\mathbb{1}_{\mathrm{out}}\rangle\rangle\langle\langle \mathbb{1}_{\mathrm{out}}|) = R_{\mathcal{M}}, \tag{7}$$

$$\mathcal{M}^* = (\mathcal{M}^\dagger)^\tau, \tag{8}$$

where $\mathcal{M}^\dagger$ is the adjoint of the linear operator $\mathcal{M}$ with respect to the Hilbert–Schmidt scalar product, i.e., $\langle B, \mathcal{M}(A)\rangle_{\mathrm{HS}} = \langle \mathcal{M}^\dagger(B), A\rangle_{\mathrm{HS}}$. According to the above definitions, one has the useful relation

$$R_{\mathcal{B}\mathcal{M}\mathcal{A}} = (\mathcal{B} \otimes \mathcal{A}^\tau)(R_{\mathcal{M}}), \tag{9}$$

where $\mathcal{B}$ is an arbitrary map from $\mathsf{Lin}(\mathsf{H}_{\mathrm{out}})$ to $\mathsf{Lin}(\mathsf{H}_{\mathrm{out}})$ and $\mathcal{A}$ is an arbitrary map from $\mathsf{Lin}(\mathsf{H}_{\mathrm{in}})$ to $\mathsf{Lin}(\mathsf{H}_{\mathrm{in}})$.

It is easy to check that the linear map $\mathcal{M}$ is CP if and only if the CJ operator $R_{\mathcal{M}}$ is positive and the correspondence $\mathcal{M} \leftrightarrow R_{\mathcal{M}}$ is an isomorphism of positive cones. Moreover, $\mathcal{M}$ is trace nonincreasing if and only if the following dominance relation holds:

$$\mathrm{Tr}_{\mathrm{out}}[R_{\mathcal{M}}] \leq \mathbb{1}_{\mathrm{in}}, \tag{10}$$

the equal sign corresponding to the trace-preserving case of the quantum channel.

Another convenient isomorphism is the one between linear maps $\mathcal{M}$ from $\mathsf{Lin}(\mathsf{H}_{\mathrm{in}})$ to $\mathsf{Lin}(\mathsf{H}_{\mathrm{out}})$ and linear operators $\check{R}_{\mathcal{M}}$ from $\mathsf{H}_{\mathrm{in}}^{\otimes 2}$ to $\mathsf{H}_{\mathrm{out}}^{\otimes 2}$ given by

$$\check{R}_{\mathcal{M}} | A\rangle\rangle = | \mathcal{M}(A)\rangle\rangle \quad \forall A \in \mathsf{Lin}(\mathsf{H}_{\mathrm{in}}), \tag{11}$$

such a definition depending on the two chosen basis $\{|c_m\rangle\}$ and $\{|e_n\rangle\}$ for $\mathsf{H}_{\mathrm{out}}$ and H_{in}, respectively. In this case one has $\check{R}_{\mathcal{A}\mathcal{B}} = \check{R}_{\mathcal{A}}\check{R}_{\mathcal{B}}$, namely, the correspondence $\mathcal{M} \leftrightarrow \check{R}_{\mathcal{M}}$ is an isomorphism of (finite dimensional) algebras. The correspondence $\mathcal{M} \leftrightarrow \check{R}_{\mathcal{M}}$ also induces a one-to-one correspondence between $R_{\mathcal{M}} \in \mathsf{Lin}(\mathsf{H}_{\mathrm{out}} \otimes \mathsf{H}_{\mathrm{in}})$ and $\check{R}_{\mathcal{M}} \in \mathsf{Lin}(\mathsf{H}_{\mathrm{in}}^{\otimes 2}, \mathsf{H}_{\mathrm{out}}^{\otimes 2})$,

$$\check{R}_{\mathcal{M}} = \mathbf{I}(R_{\mathcal{M}}). \tag{12}$$

Like $\check{R}_{\mathcal{M}}$, the isomorphism $\mathbf{I}$ depends on the two chosen basis $\{|c_m\rangle\}$ and $\{|e_n\rangle\}$ for $\mathsf{H}_{\mathrm{out}}$ and H_{in}.

Every quantum operation $\mathcal{M}$ can be written in a (nonunique) Kraus form,

$$\mathcal{M}(\rho) = \sum_i M_i \rho M_i^\dagger \quad \forall \rho \in \mathsf{Lin}(\mathsf{H}_{\mathrm{in}}). \tag{13}$$

Any Kraus form is equivalent to a decomposition of the CJ positive operator $R_{\mathcal{M}}$ into rank-one positive operators,

$$R_{\mathcal{M}} = \sum_i | M_i\rangle\rangle\langle\langle M_i |. \tag{14}$$

In particular, diagonalization of $R_{\mathcal{M}}$ yields the canonical Kraus form $\mathcal{M}(\rho) = \Sigma_i K_i \rho K_i^\dagger$, $\mathrm{Tr}[K_i^\dagger K_j] = \delta_{ij}\|K_i\|_2^2$, where $\|X\|_2 := \mathrm{Tr}[X^\dagger X]^{1/2}$ is the Hilbert–Schmidt norm.

For a map $\mathcal{M}$ with Kraus form (13), it is immediate to show that the maps $\mathcal{M}^\dagger$, $\mathcal{M}^\tau$, and $\mathcal{M}^*$ have the Kraus forms

$$\mathcal{M}^*(\rho) = \sum\nolimits_i M_i^* \rho M_i^\tau \quad \forall \rho \in \mathsf{Lin}(\mathsf{H}_{\mathrm{in}}),$$

$$\mathcal{M}^\dagger(A) = \sum\nolimits_i M_i^\dagger A M_i \quad \forall A \in \mathsf{Lin}(\mathsf{H}_{\mathrm{out}}),$$

$$\mathcal{M}^\tau(A) = \sum\nolimits_i M_i^\tau A M_i^* \quad \forall A \in \mathsf{Lin}(\mathsf{H}_{\mathrm{out}}). \tag{15}$$

Moreover, using Eqs. (11) and (5) the operator $\check{R}_{\mathcal{M}} = \mathbf{I}(R_{\mathcal{M}})$ can be written in terms of the Kraus form as follows:

$$\mathbf{I}(R_{\mathcal{M}}) = \sum_i M_i \otimes M_i^*. \tag{16}$$

Different Kraus decompositions are all connected to a minimal one—e.g., the canonical—by an isometric matrix W as follows:

$$M_i = \sum_j W_{ij} K_j, \quad \sum_k W_{ki}^* W_{kj} = \delta_{ij}. \tag{17}$$

Therefore, the operator space $\mathsf{Span}\{M_i\}$ is independent of the choice of the Kraus form and is a function only of the map $\mathcal{M}$. In the following we will make use of the corresponding Hilbert space, which is spanned by the bipartite vectors $\{| M_i\rangle\rangle\}$:

$$\mathsf{H}_{\mathcal{M}} = \mathsf{Span}\{| M_i\rangle\rangle\} \equiv \mathsf{Supp}(R_{\mathcal{M}}) \equiv \mathsf{Rng}(R_{\mathcal{M}}), \tag{18}$$

having used that the CJ operator $R_{\mathcal{M}}$ is positive, whence support and range coincide. Note that generally $\mathsf{H}_{\mathcal{M}}$ can be a proper subspace of $\mathsf{H}_{\mathrm{out}} \otimes \mathsf{H}_{\mathrm{in}}$.

C. Operator frames

The Kraus operators $\{M_i\}$ are generally nonorthogonal and not even linearly independent. They are a so-called operator frame for the operator space $\mathsf{Span}\{M_i\}$, namely, a (possibly infinite) set of vectors such that the sum $\Sigma_i | M_i\rangle\rangle\langle\langle M_i |$ converges to an operator $M \in \mathsf{Lin}(\mathsf{H}_{\mathrm{out}} \otimes \mathsf{H}_{\mathrm{in}})$, called the frame operator. For Kraus operators, we have indeed

$$M := \sum_i | M_i\rangle\rangle\langle\langle M_i | = R_{\mathcal{M}}. \tag{19}$$

Any vector $| A\rangle\rangle$ in $\mathsf{Supp}(R_{\mathcal{M}})$ can be expanded on the frame $\{| M_i\rangle\rangle\}$, and the expansion can be written in terms of another set of operators $\{N_i\}$ called dual of the frame $\{M_i\}$ as $| A\rangle\rangle = \Sigma_i | M_i\rangle\rangle\langle\langle N_i | A\rangle\rangle$. Equivalently, we have the completeness relation

$$\sum_i | M_i\rangle\rangle\langle\langle N_i | = \mathbb{1}_{\mathrm{out}} \otimes \mathbb{1}_{\mathrm{in}}. \tag{20}$$

A possible choice of dual, particularly relevant to our purposes, is the canonical dual given by

$$| \hat{M}_i\rangle\rangle = M^{-1} | M_i\rangle\rangle. \tag{21}$$

The inverse M^{-1} is actually defined on $\mathsf{H}_{\mathcal{M}} = \mathsf{Rng}(R_{\mathcal{M}}) \equiv \mathsf{Supp}(R_{\mathcal{M}})$, and on $\mathsf{H}_{\mathrm{out}} \otimes \mathsf{H}_{\mathrm{in}}$, it must be regarded as the Moore–Penrose pseudoinverse with support on $\mathsf{H}_{\mathcal{M}}$.

D. Minimal dilation of a quantum operation

For $\{M_i\}$ Kraus operators of the CP map $\mathcal{M}$, the frame operator M is just the CJ operator $R_{\mathcal{M}} \equiv M$ of the map $\mathcal{M}$, whence it is independent of the choice of the Kraus operators $\{M_i\}$. Consider now the operator $V: \mathsf{H}_{\mathrm{in}} \to \mathsf{H}_{\mathrm{out}} \otimes \mathsf{H}_{\mathcal{M}}$ defined by

$$V := \sum_i M_i \otimes ((M^{\tau})^{-1/2} | M_i^*\rangle\rangle) = \sum_i M_i \otimes ((M^*)^{-1/2} | M_i^*\rangle\rangle), \tag{22}$$

having used $M^* = M^{\tau}$ since $M \geq 0$. Note that V is independent of the choice of the Kraus form: indeed, one has

$$V = \sum_{ijk} W_{ij} W_{ik}^* K_j \otimes ((R_{\mathcal{M}}^*)^{-1/2} | K_k^* \rangle\rangle) = \sum_j K_j \otimes ((R_{\mathcal{M}}^*)^{-1/2} | K_j^* \rangle\rangle) = \sum_j K_j \otimes \frac{| K_j^* \rangle\rangle}{\|K_j\|_2}, \qquad (23)$$

having used Eq. (17) and the fact that each canonical Kraus operator K_i is eigenvector of $R_{\mathcal{M}}$ with eigenvalue $\|K_i\|_2^2$. Clearly the operator V provides a dilation of the CP map $\mathcal{M}$, with $\mathsf{H}_{\mathcal{M}}$ playing the role of ancillary Hilbert space,

$$\mathcal{M}(\rho) = \mathrm{Tr}_{\mathsf{H}_{\mathcal{M}}}[V\rho V^\dagger]. \qquad (24)$$

For quantum operations V is a contraction ($V^\dagger V \le \mathbb{1}_{\mathrm{in}}$), while for quantum channels V is an isometry ($V^\dagger V = \mathbb{1}_{\mathrm{in}}$).

Among all possible dilations of $\mathcal{M}$, the one given by V in Eq. (24) has minimum ancilla dimension. Indeed, for any operator $V' : \mathsf{H}_{\mathrm{in}} \to \mathsf{H}_{\mathrm{in}} \otimes \mathsf{H}_A$, such that $\mathcal{M}(\rho) = \mathrm{Tr}_A[V'\rho V'^\dagger]$, the map $\mathcal{M}(\rho)$ has Kraus representation $\{M_i := {}_A\langle i | V\}$, where $\{|i\rangle_A\}$ is an orthonormal basis for H_A. Then, according to Eq. (14) $\dim(\mathsf{H}_A) \ge \mathrm{rank}(R_{\mathcal{M}}) = \dim(\mathsf{H}_{\mathcal{M}})$. In other words, V is the minimal Stinespring dilation of the CP map $\mathcal{M}$.[19] Any nonminimal dilation V' is connected to the minimal one via an isometry of ancillary spaces $Y : \mathsf{H}_{\mathcal{M}} \to \mathsf{H}_A$, $Y^\dagger Y = \mathbb{1}_{\mathsf{H}_{\mathcal{M}}}$. Indeed, using Eq. (17), one has

$$V' = \sum_{ij} W_{ij} K_j \otimes |i\rangle = \sum_j K_j \otimes |\psi_j\rangle = (\mathbb{1}_{\mathrm{out}} \otimes Y)V, \qquad (25)$$

where $\{|\psi_j\rangle \in \mathsf{H}_A\}$ are the orthonormal vectors $|\psi_j\rangle := \Sigma_i W_{ij} |i\rangle$ and Y is the isometry $Y := |\psi_j\rangle\langle\langle K_j^* | / \|K_j\|_2$. The minimal Stinespring dilation is unique up to local unitaries on the ancilla Hilbert space $\mathsf{H}_{\mathcal{M}}$, namely, if V' is also minimal, then Y is a unitary from $\mathsf{H}_{\mathcal{M}}$ to H_A.[19]

We now give a new compact formula for the minimal Stinespring dilation of a CP map in terms of the CJ operator $R_{\mathcal{M}}$.

Proposition 1: *Let $R_{\mathcal{M}} \in \mathsf{Lin}(\mathsf{H}_{\mathrm{out}}, \mathsf{H}_{\mathrm{in}})$ be the CJ operator associated with the CP map $\mathcal{M} \in \mathsf{CP}(\mathsf{H}_{\mathrm{in}}, \mathsf{H}_{\mathrm{out}})$, and let $\mathsf{H}_{\mathcal{M}}$ be the Hilbert space $\mathsf{H}_{\mathcal{M}} = \mathsf{Supp}(R_{\mathcal{M}}) = \mathsf{Rng}(R_{\mathcal{M}})$. Then, a minimal dilation $V : \mathsf{H}_{\mathrm{in}} \to \mathsf{H}_{\mathrm{out}} \otimes \mathsf{H}_{\mathcal{M}}$ is given by*

$$V = (\mathbb{1}_{\mathrm{out}} \otimes (R_{\mathcal{M}}^\tau)^{1/2})(|\mathbb{1}_{\mathrm{out}}\rangle\rangle \otimes \mathbb{1}_{\mathrm{in}}), \qquad (26)$$

or, alternatively, by

$$V = (\mathbf{I}(R_{\mathcal{M}}^{1/2}) \otimes \mathbb{1}_{\mathrm{in}})(\mathbb{1}_{\mathrm{in}} \otimes | \mathbb{1}_{\mathrm{in}}\rangle\rangle), \qquad (27)$$

$\mathbf{I}$ *being the one-to-one correspondence defined in Eq. (12)*

Proof: It is simple to check that Eq. (26) provides a dilation of $\mathcal{M}$, which is clearly minimal since the ancilla space is $\mathsf{H}_{\mathcal{M}} = \mathsf{Supp}(R_{\mathcal{M}})$. Indeed, using the inclusion $\mathsf{H}_{\mathrm{out}} \otimes \mathsf{H}_{\mathcal{M}} \subseteq \mathsf{H}_{\mathrm{out}}^{\otimes 2} \otimes \mathsf{H}_{\mathrm{in}}$ and Eqs. (5) and (6), one has for any $\rho \in \mathsf{S}(\mathsf{H}_{\mathrm{in}})$

$$\begin{aligned}\mathrm{Tr}_{\mathsf{H}_{\mathcal{M}}}[V\rho V^\dagger] &= \mathrm{Tr}_{\mathrm{out}_2}\mathrm{Tr}_{\mathrm{in}}[(\mathbb{1}_{\mathrm{out}} \otimes R_{\mathcal{M}}^\tau)(| \mathbb{1}_{\mathrm{out}}\rangle\rangle\langle\langle \mathbb{1}_{\mathrm{out}} | \otimes \rho)] = \mathrm{Tr}_{\mathrm{out}_2}[(\mathbb{1}_{\mathrm{out}} \otimes \mathrm{Tr}_{\mathrm{in}}[(\mathbb{1}_{\mathrm{out}} \otimes \rho)R_{\mathcal{M}}^\tau]) \\ &\quad \times (| \mathbb{1}_{\mathrm{out}}\rangle\rangle\langle\langle \mathbb{1}_{\mathrm{out}} |)] = \mathrm{Tr}_{\mathrm{out}_2}[(\mathbb{1}_{\mathrm{out}} \otimes \mathcal{M}(\rho)^\tau)| \mathbb{1}_{\mathrm{out}}\rangle\rangle\langle\langle \mathbb{1}_{\mathrm{out}} |] = \mathcal{M}(\rho),\end{aligned} \qquad (28)$$

where $\mathrm{Tr}_{\mathrm{out}_2}$ denoting partial trace over the second copy of $\mathsf{H}_{\mathrm{out}}$ in the tensor product $\mathsf{H}_{\mathrm{out}}^{\otimes 2} \otimes \mathsf{H}_{\mathrm{in}}$. On the other hand, using the relation $(R_{\mathcal{M}})^s = \Sigma_j \|K_j\|^{2(s-1)} | K_j\rangle\rangle\langle\langle K_j |$ along with Eq. (16), we get $\mathbf{I}(R_{\mathcal{M}}^{1/2}) = \Sigma_j \|K_j\|^{-1/2} K_j \otimes K_j^*$. Substituting $\mathbf{I}(R_{\mathcal{M}}^{1/2})$ into Eq. (27) and comparing with Eq. (23), we obtain that the definitions of V in Eqs. (27) and (22) actually coincide. Finally, direct calculation shows the coincidence of definitions of V in (26) and (27). □

E. POVMs and quantum Instruments

The statistics of a quantum measurement is described by a measurable space Ω with a σ-algebra Σ_Ω of events and a probability measure p on (Ω, Σ_Ω). In quantum mechanics the probability measure in terms of the quantum state ρ is given by the Born rule,

$$\forall B \in \Sigma_\Omega, \quad p(B) = \mathrm{Tr}[\rho P_B], \tag{29}$$

where P is a POVM, namely, a map from events $B \in \Sigma_\Omega$ to positive operators $P_B \geq 0$ on H, satisfying the requirements

$$P_\Omega = \mathbb{1} \quad \text{(normalization)}, \tag{30}$$

$$P_{(\cup_{i=1}^\infty B_i)} = \sum_{i=1}^\infty P_{B_i} \quad \forall \{B_i\}: B_i \cap B_j = \emptyset \ \forall i \neq j \quad (\sigma\text{-additivity}), \tag{31}$$

where the series converges in the weak operator topology.

A complete description of a measurement in a cascade of different measurements performed on the same system must also provide the conditional state associated with any possible event. In quantum mechanics this is given by the notion of "quantum instrument,"[2] in which each event $B \in \Sigma_\Omega$ corresponds to a quantum operation $\mathcal{Z}_B \in \mathsf{CP}(\mathsf{H}_{\mathrm{in}}, \mathsf{H}_{\mathrm{out}})$. More precisely, we have the following.

Definition 1: *A map* $\mathcal{Z}: \Sigma_\Omega \to \mathsf{CP}(\mathsf{H}_{\mathrm{in}}, \mathsf{H}_{\mathrm{out}})$ *is a* quantum instrument *if it satisfies the properties*

$$\mathrm{Tr}[\mathcal{Z}_\Omega(\rho)] = \mathrm{Tr}[\rho] \quad \forall \rho \in \mathsf{S}(\mathsf{H}_{\mathrm{in}}), \tag{32}$$

$$\mathcal{Z}_{(\cup_{i=1}^\infty B_i)} = \sum_{i=1}^\infty \mathcal{Z}_{B_i} \quad \forall \{B_i\}: B_i \cap B_j = \emptyset \quad \forall i \neq j. \tag{33}$$

Using the CJ isomorphism (6), any instrument $\mathcal{Z}$ can be associated in a one-to-one fashion with a POVM Z, which we call CJ measure (CJM) of the instrument, given by

$$Z_B := R_{\mathcal{Z}_B} = (\mathcal{Z}_B \otimes \mathcal{I})(|\mathbb{1}_{\mathrm{in}}\rangle\rangle\langle\langle \mathbb{1}_{\mathrm{in}}|) \quad \forall B \in \Sigma_\Omega. \tag{34}$$

Differently from usual POVMs, for which the normalization is given by Eq. (30) the measure Z has the normalization condition,

$$\mathrm{Tr}_{\mathrm{out}}[Z_\Omega] = \mathbb{1}_{\mathrm{in}}, \tag{35}$$

$\mathrm{Tr}_{\mathrm{out}}[\cdot]$ denoting partial trace over $\mathsf{H}_{\mathrm{out}}$.

The POVM P giving the probability of the event $B \in \Sigma_\Omega$ for state $\rho \in \mathsf{S}(\mathsf{H}_{\mathrm{in}})$ can be written in terms of the CJM Z using the isomorphism (6) as follows:

$$\mathrm{Tr}[P_B \rho] = \mathrm{Tr}[\mathcal{Z}_B(\rho)] = \mathrm{Tr}[(\mathbb{1}_{\mathrm{out}} \otimes \rho^\tau) Z_B] = \mathrm{Tr}[\rho \mathrm{Tr}_{\mathrm{out}}[Z_B^\tau]], \tag{36}$$

whence

$$P_B = \mathrm{Tr}_{\mathrm{out}}[Z_B^\tau] \quad \forall B \in \Sigma_\Omega. \tag{37}$$

In finite dimensions, the correspondence between instruments and CJMs allows one to simply prove the existence of an instrument density with respect to a suitable scalar measure.[20,21]

Proposition 2: *Any instrument* $\mathcal{Z}: \Sigma_\Omega \to \mathsf{CP}(\mathsf{H}_{\mathrm{in}}, \mathsf{H}_{\mathrm{out}})$ *in finite dimensions can be written as*

$$\mathcal{Z}_B = \int_B \mu(d\omega) \mathcal{S}_\omega, \tag{38}$$

where $\mu(d\omega)$ *is the finite measure defined by* $\mu(B) \doteq \mathrm{Tr}[Z_B]$ $\forall B \in \Sigma_\Omega$, *and the density* $\mathcal{S}_\omega$ *is a* CP-*map valued function, uniquely defined* μ-*almost everywhere.*

Proof: Let $\{|k\rangle\}$ be an orthonormal basis for $\mathsf{H}_{\mathrm{out}} \otimes \mathsf{H}_{\mathrm{in}}$ and define the complex measures μ_{kl}, $\mu_{kl}(B) := \langle k|Z_B|l\rangle$. Due to positivity, one has $|\mu_{kl}(B)| \leq \sqrt{\langle k|Z_B|k\rangle\langle l|Z_B|l\rangle} \leq \mathrm{Tr}[Z_B] = \mu(B)$, i.e., all measures μ_{kl} are absolutely continuous with respect to μ. Therefore, any measure μ_{kl} admits a density $\sigma_{kl}(\omega)$ with respect to μ. We then have

$$Z_B = \sum_{k,l} \mu_{kl}(B)|k\rangle\langle l| = \sum_{k,l} \int_B \mu(d\omega)\sigma_{kl}(\omega)|k\rangle\langle l| = \int_B \mu(d\omega)S_\omega, \tag{39}$$

having defined $S_\omega := \Sigma_{k,l}\sigma_{kl}(\omega)|k\rangle\langle l|$. Since S_ω is the density of the POVM Z_B with respect to the scalar measure μ, it is positive and uniquely defined μ-a.e. The instrument density $\mathcal{S}_\omega$ is then obtained by the relation $\mathcal{S}_\omega(\rho) = \mathrm{Tr}_{\mathrm{in}}[(\mathbb{1}_{\mathrm{out}} \otimes \rho^\tau)S_\omega]$. □

III. DILATIONS OF QUANTUM INSTRUMENTS

We are now in position to prove a dilation theorem for instruments with generally continuous-outcome space in finite dimensions. A dilation of a quantum instrument $\mathcal{Z}: \Sigma_\Omega \to \mathsf{CP}(\mathsf{H}_{\mathrm{in}}, \mathsf{H}_{\mathrm{out}})$ is a triple (H_A, V, Q), where H_A is an ancillary Hilbert space, $V: \mathsf{H}_{\mathrm{in}} \to \mathsf{H}_{\mathrm{out}} \otimes \mathsf{H}_A$ is an isometry, and $Q: \Sigma_\Omega \to \mathsf{Lin}(\mathsf{H}_A)$ is a POVM on the ancilla, such that

$$\mathcal{Z}_B(\rho) = \mathrm{Tr}_{\mathsf{H}_A}[V\rho V^\dagger(\mathbb{1}_{\mathrm{out}} \otimes Q_B)] \quad \forall\, B \in \Sigma_\Omega. \tag{40}$$

The triple (H_A, V, Q) represents an indirect measurement scheme where the input system H_{in} evolves through the isometry V, producing the output $\mathsf{H}_{\mathrm{out}}$ and the ancilla H_A, which then undergoes a POVM measurement Q with the same outcome space as the instrument.

Theorem 1: *Let* $\mathcal{Z}: \Sigma_\Omega \to \mathsf{CP}(\mathsf{H}_{\mathrm{in}}, \mathsf{H}_{\mathrm{out}})$ *be an instrument with outcome space* Ω *and* $Z: \Sigma_\Omega \to \mathsf{Lin}(\mathsf{H}_{\mathrm{out}} \otimes \mathsf{H}_{\mathrm{in}})$ *be the associated CJM. A minimal dilation of the instrument is given by the triple* (H_A, V, Q) *where the ancillary Hilbert space* H_A *is isomorphic to* $\mathsf{H}_{\mathcal{Z}} := \mathsf{Supp}(Z_\Omega) = \mathsf{Rng}(Z_\Omega)$, $V: \mathsf{H}_{\mathrm{in}} \to \mathsf{H}_{\mathrm{out}} \otimes \mathsf{H}_A$ *is the isometry*

$$V := (\mathbb{1}_{\mathrm{out}} \otimes (Z_\Omega^\tau)^{1/2})(|\,\mathbb{1}_{\mathrm{out}}\rangle\rangle \otimes \mathbb{1}_{\mathrm{in}}), \tag{41}$$

and Q *is the POVM on* H_A *given by*

$$Q_B := (Z_\Omega^{-1/2} Z_B Z_\Omega^{-1/2})^\tau \quad \forall\, B \in \Sigma_\Omega. \tag{42}$$

Proof: According to Eq. (26), V is the minimal Stinespring isometry of the channel $\mathcal{Z}_\Omega$. On the other hand, Q is clearly a POVM on H_A, since $Q_B \geq 0 \;\; \forall B \in \Sigma_\Omega$ and $Q_\Omega = \mathbb{1}_{\mathsf{H}_A}$. Moreover, exploiting the inclusion $\mathsf{H}_{\mathrm{out}} \otimes \mathsf{H}_A \subseteq \mathsf{H}_{\mathrm{out}}^{\otimes 2} \otimes \mathsf{H}_{\mathrm{in}}$, we have

$$\begin{aligned}\mathrm{Tr}_{\mathsf{H}_A}[V\rho V^\dagger(\mathbb{1}_{\mathrm{out}} \otimes Q_B)] &= \mathrm{Tr}_{\mathrm{out}_2}\mathrm{Tr}_{\mathrm{in}}[(|\,\mathbb{1}_{\mathrm{out}}\rangle\rangle\langle\langle \mathbb{1}_{\mathrm{out}}\,| \otimes \rho)(\mathbb{1}_{\mathrm{out}} \otimes Z_B^\tau)] \\ &= \mathrm{Tr}_{\mathrm{out}_2}[(\mathbb{1}_{\mathrm{out}} \otimes \mathcal{Z}_B(\rho)^\tau)|\,\mathbb{1}_{\mathrm{out}}\rangle\rangle\langle\langle \mathbb{1}_{\mathrm{out}}\,|] = \mathcal{Z}_B(\rho),\end{aligned} \tag{43}$$

thus proving that (H_A, V, Q) is actually a dilation of the instrument $\mathcal{Z}$. Finally, the dilation has minimal ancilla dimension. Indeed, for any dilation $(\mathsf{H}_{A'}, V', Q')$ of the instrument $\mathcal{Z}$, we have a dilation of the channel $\mathcal{Z}_\Omega$, given by $\mathcal{Z}_\Omega(\rho) = \mathrm{Tr}_{\mathsf{H}_{A'}}[V'\rho V'^\dagger]$. Since V is the minimal Stinespring isometry of the channel $\mathcal{Z}_\Omega$, one necessarily has $\dim(\mathsf{H}_{A'}) \geq \dim(\mathsf{Supp}(Z_\Omega)) \equiv \dim(\mathsf{H}_{\mathcal{A}})$. □

Any other dilation of the instrument $\mathcal{Z}$ arises from some nonminimal isometry $V': \mathsf{H}_{\mathrm{in}} \to \mathsf{H}_{\mathrm{out}} \otimes \mathsf{H}_A$, which is necessarily of the form $V' = (\mathbb{1}_{\mathrm{out}} \otimes Y)V$, where $Y: \mathsf{H}_{\mathcal{A}} \to \mathsf{H}_{A'}$ is an isometry of ancilla spaces. Substituting the form of V' in Eq. (40), we then obtain

$$Q_B = Y^\dagger Q'_B Y \quad \forall\, B \in \Sigma_\Omega. \tag{44}$$

Since Y can be viewed as an isometric embedding of $\mathsf{H}_{\mathcal{A}}$ into $\mathsf{H}_{A'}$, the above equation means that Q is the projection of Q' on the support of Y. This is indeed the case of the nonminimal dilation provided by the dilation theorem of Ozawa,[2] where Q' is a PVM on the infinite-dimensional ancilla space $\mathsf{H}_{A'}$. According to Eq. (44), Q' is then a Naimark dilation[8] of the minimal POVM Q provided in our theorem.

IV. INSTRUMENTS GENERATED BY OPERATOR FRAMES

We now introduce the definition of frame-orbit instruments, which will play an important role in the construction of feed-forward realization schemes and generalized teleportation schemes.

Fix a finite measure $\mu(d\omega)$ on the outcome space Ω, a measurable family of quantum operations $\mathcal{A}_\Omega := \{\mathcal{A}_\omega\}_{\omega\in\Omega}$, and a measurable family of quantum channels $\mathcal{B}_\Omega := \{\mathcal{B}_\omega\}_{\omega\in\Omega}$. Then we have the following.

Definition 2: (Frame-orbit instruments) *An instrument $\mathcal{Z}:\Sigma_\Omega \to \mathsf{CP}(\mathsf{H}_{\mathrm{in}},\mathsf{H}_{\mathrm{out}})$ is a frame-orbit instrument with respect to $(\mu,\mathcal{A}_\Omega,\mathcal{B}_\Omega)$ if $\mathcal{Z}$ admits a density with respect to μ, and the density has the form*

$$\mathcal{S}_\omega = \mathcal{B}_\omega \mathcal{S}_0 \mathcal{A}_\omega^\dagger \quad \mu - \forall\ \omega \in \Omega, \tag{45}$$

where $\mathcal{S}_0$ is a fixed CP map. In the case $\mathcal{B}_\omega \equiv \mathcal{I}_{\mathrm{out}} \forall\, \omega \in \Omega$ we say that $\mathcal{Z}$ is a frame-orbit instrument with respect to $(\mu,\mathcal{A}_\Omega)$.

According to Proposition 2, any instrument $\mathcal{Z}$ can be trivially viewed as a frame-orbit instrument by taking $\mu(B) := \mathrm{Tr}[Z_B]$, $\mathcal{A}_\omega \equiv \mathcal{S}_\omega$, $\mathcal{S}_0 \equiv \mathcal{I}$, and $\mathcal{B}_\omega \equiv \mathcal{I}$. However, a given instrument can be a frame-orbit instrument with respect to several different triples $(\mu,\mathcal{A}_\Omega,\mathcal{B}_\Omega)$, and, on the contrary, once a triple $(\mu,\mathcal{A}_\Omega,\mathcal{B}_\Omega)$ has been fixed, not all instruments are frame-orbit instruments with respect to that triple.

From now on we will restrict our attention to the case where the elements of $\mathcal{A}_\Omega$ are single-Kraus operations $\mathcal{A}_\omega(\cdot) := A_\omega \cdot A_\omega^\dagger$. The generalization of all results to the case $\mathcal{A}_\omega(\cdot) = \sum_{k=1}^{d_{\mathrm{in}}^2} A_{\omega,k} \cdot A_{\omega,k}^\dagger$ is straightforward, as it only consists in replacing the index ω by the couple (ω,k), $\int_\Omega \mu(d\omega)$ by $\int_\Omega \mu(d\omega)\sum_{k=1}^{d_{\mathrm{in}}^2}$, and taking $\mathcal{B}_{(\omega,k)} := \mathcal{B}_\omega$ and $\mathcal{S}_\omega := \sum_{k=1}^{d_{\mathrm{in}}^2} \mathcal{S}_{(\omega,k)}$.

Lemma 1: *Let $\mathcal{Z}:\Sigma_\Omega \to \mathsf{CP}(\mathsf{H}_{\mathrm{in}},\mathsf{H}_{\mathrm{out}})$ be a frame-orbit instrument with respect to $(\mu,\mathcal{A}_\Omega,\mathcal{B}_\Omega)$ with density $\mathcal{S}_\omega = \mathcal{B}_\omega \mathcal{S}_0 \mathcal{A}_\omega^\dagger$, $\mathcal{S}_0(\cdot) = \sum_{i=1}^r S_i \cdot S_i^\dagger$ be a Kraus form for $\mathcal{S}_0$ and $\xi := \sum_{i=0}^r S_i^\dagger S_i \in \mathsf{Lin}(\mathsf{H}_{\mathrm{in}})$. Then,*

$$\int_\Omega \mu(d\omega) A_\omega \xi A_\omega^\dagger = \mathrm{Tr}_{\mathrm{in}_2}[(\mathbb{1}_{\mathrm{in}} \otimes \xi^\tau)A] = \mathbb{1}_{\mathrm{in}}, \tag{46}$$

where $A \in \mathsf{Lin}(\mathsf{H}_{\mathrm{in}}^{\otimes 2})$ is the frame operator,

$$A = \int_\Omega \mu(d\omega) | A_\omega \rangle\rangle\langle\langle A_\omega |, \tag{47}$$

and $\mathrm{Tr}_{\mathrm{in}_2}$ denotes partial trace over the second copy of H_{in} in the tensor $\mathsf{H}_{\mathrm{in}}^{\otimes 2}$. Vice versa, for any positive operator $\xi \in \mathsf{Lin}(\mathsf{H}_{\mathrm{in}})$ satisfying Eq. (46) there exists a frame-orbit instrument with respect to $(\mu,\mathcal{A}_\Omega,\mathcal{B}_\Omega)$.

Proof: For the normalization of the instrument $\mathcal{Z}$, $\mathcal{Z}_\Omega$ must be trace preserving, and we have $\mathrm{Tr}[\mathcal{Z}_\Omega(\rho)] = \int_\Omega \mu(d\omega)\mathrm{Tr}[A_\omega \xi A_\omega^\dagger \rho] = \mathrm{Tr}[\rho]\, \forall\, \rho \in \mathsf{S}(\mathsf{H}_{\mathrm{in}})$, whence Eq. (46). Vice versa, for any $\xi \geq 0$ satisfying Eq. (46), we can define $\mathcal{S}_0(\cdot) := \xi^{1/2} \cdot \xi^{1/2}$, so that $\mathcal{S}_\omega := \mathcal{B}_\omega \mathcal{S}_0 \mathcal{A}_\omega^\dagger$ is the density of a normalized frame-orbit instrument. □

In particular, whenever the elements of $\mathcal{A}_\Omega$ are all proportional to unitary channels, the class of frame-orbit instruments with respect to $(\mu,\mathcal{A}_\Omega,\mathcal{B}_\Omega)$ is nonempty, as one can choose, e.g., $\xi = \kappa \mathbb{1}_{\mathrm{in}}$ with suitable normalization constant $\kappa > 0$. As we will see in Sec. V, this includes the case of group-covariant instruments. Similarly, if $\mathsf{A}_\Omega := \{A_\omega\}$ is a tight operator frame, namely, $A = \mathbb{1}_{\mathrm{in}}^{\otimes 2}$, by definition Eq. (46) holds for any $\xi \in \mathsf{Lin}(\mathsf{H}_{\mathrm{in}})$. Notice that, however, the operators in A_Ω do not need to be unitary, in general, nor A_Ω needs to be a tight frame, since it is enough that Eq. (46) holds for a single operator $0 \leq \xi \in \mathsf{Lin}(\mathsf{H}_{\mathrm{in}})$.

A. Canonically associated POVMs and their densities

According to Sec. III, there are two POVMs P and Q that are canonically associated with the instrument $\mathcal{Z}$. The POVM P gives the probability distribution of the instrument for each event and each state, whereas the POVM Q allows to express the minimal dilation of the instrument via the minimal isometry $V=(\mathbb{1}_{\mathrm{out}}\otimes Z_{\Omega}^{\tau 1/2})(|\mathbb{1}_{\mathrm{out}}\rangle\rangle\otimes\mathbb{1}_{\mathrm{in}})$. Both P and Q can be written in terms of the CJM $Z_B=\mathcal{Z}_B\otimes\mathcal{I}(|\mathbb{1}_{\mathrm{in}}\rangle\rangle\langle\langle\mathbb{1}_{\mathrm{in}}|)$ of the instrument [see Eq. (34)] as

$$P_B=\mathrm{Tr}_{\mathrm{out}}[Z_B^{\tau}],\quad Q_B=(Z_{\Omega}^{-1/2}Z_BZ_{\Omega}^{-1/2})^{\tau}. \tag{48}$$

Obviously, since the instrument $\mathcal{Z}$ admits a density with respect to μ, also the CJM Z will admit a density with respect to μ, given by

$$S_{\omega}\coloneqq(\mathcal{S}_{\omega}\otimes\mathcal{I})(|\mathbb{1}_{\mathrm{in}}\rangle\rangle\langle\langle\mathbb{1}_{\mathrm{in}}|), \tag{49}$$

which is positive and uniquely defined μ-almost everywhere. From Eqs. (45), (9), and (8), it is also clear that the density S_{ω} has the form

$$S_{\omega}=(\mathcal{B}_{\omega}\otimes\mathcal{A}_{\omega}^{*})(S_0), \tag{50}$$

having defined $S_0\coloneqq(\mathcal{S}_0\otimes\mathcal{I})(|\mathbb{1}_{\mathrm{in}}\rangle\rangle\langle\langle\mathbb{1}_{\mathrm{in}}|)=\Sigma_{i=1}^{r}|S_i\rangle\rangle\langle\langle S_i|$. Finally, from Eq. (48) it follows that the POVMs P and Q admit densities with respect to μ, ξ_{ω}, and ζ_{ω}, respectively, given by

$$\xi_{\omega}=A_{\omega}\xi A_{\omega}^{\dagger},\quad \xi\coloneqq\mathrm{Tr}_{\mathrm{out}}[S_0^{\tau}]=\sum_{i=1}^{r}S_i^{\dagger}S_i,$$

$$\zeta_{\omega}=(Z_{\Omega}^{-1/2}S_{\omega}Z_{\Omega}^{-1/2})^{\tau}. \tag{51}$$

B. Feed-forward realization of frame-orbit instruments

The realization of frame-orbit instruments with respect to $(\mu,\mathcal{A}_{\Omega},\mathcal{B}_{\Omega})$ can be always reduced to the realization of frame-orbit instruments with respect to $(\mu,\mathcal{A}_{\Omega})$, combined with a feed-forward classical communication to implement the conditional channel $\mathcal{B}_{\omega}$. Indeed, according to Eq. (45), every frame-orbit instrument $\mathcal{Z}$ with respect to $(\mu,\mathcal{A}_{\Omega},\mathcal{B}_{\Omega})$ is equivalent to the frame-orbit instrument $\mathcal{T}$ with respect to $(\mu,\mathcal{A}_{\Omega})$ given by

$$\mathcal{T}_B=\int_B\mu(d\omega)\mathcal{S}_0\mathcal{A}_{\omega}^{\dagger} \tag{52}$$

followed by the channel $\mathcal{B}_{\omega}$ depending on the outcome ω. Notice that $\mathcal{T}$ is a normalized instrument, since $\mathcal{T}$ and $\mathcal{Z}$ have the same normalization in Eq. (46).

According to Eq. (50) the CJ operator $T_{\Omega}=(\mathcal{T}_{\Omega}\otimes\mathcal{I})(|\mathbb{1}_{\mathrm{in}}\rangle\rangle\langle\langle\mathbb{1}_{\mathrm{in}}|)$ is then given by

$$T_{\Omega}=\int_{\Omega}\mu(d\omega)(\mathcal{I}\otimes\mathcal{A}_{\omega}^{*})(S_0)=(\mathcal{S}_0\otimes\mathcal{I})\left(\int_{\Omega}\mu(d\omega)(\mathcal{I}\otimes\mathcal{A}_{\omega}^{*})(|\mathbb{1}_{\mathrm{in}}\rangle\rangle\langle\langle\mathbb{1}_{\mathrm{in}}|)\right)=(\mathcal{S}_0\otimes\mathcal{I})(EA^{*}E), \tag{53}$$

with $\mathcal{S}_0\in\mathsf{CP}(\mathsf{H}_{\mathrm{in}_1},\mathsf{H}_{\mathrm{out}})$ and $A\in\mathsf{Lin}(\mathsf{H}_{\mathrm{in}}^{\otimes 2})$ being the frame operator in Eq. (47), and E denoting the unitary swap between the two copies of H_{in} in the tensor $\mathsf{H}_{\mathrm{in}}^{\otimes 2}$. Combining the feed-forward scheme with the minimal dilation of the instrument $\mathcal{T}$, we obtain the following.

Corollary 1: *Let $\mathcal{Z}\in\mathsf{CP}(\mathsf{H}_{\mathrm{in}},\mathsf{H}_{\mathrm{out}})$ be a frame-orbit instrument with respect to $(\mu,\mathcal{A}_{\Omega},\mathcal{B}_{\Omega})$ with density $\mathcal{S}_{\omega}=\mathcal{B}_{\omega}\mathcal{S}_0\mathcal{A}_{\omega}^{\dagger}$ and let $T_{\Omega}\in\mathsf{Lin}(\mathsf{H}_{\mathrm{out}}\otimes\mathsf{H}_{\mathrm{in}})$ be the CJ operator defined in Eq. (53). Then, the instrument $\mathcal{Z}$ has the minimal feed-forward realization,*

$$\mathcal{S}_{\omega}(\rho)=\mathcal{B}_{\omega}(\mathrm{Tr}_{\mathsf{H}_A}[V\rho V^{\dagger}(\mathbb{1}_{\mathrm{out}}\otimes\zeta_{\omega})]), \tag{54}$$

where V is the isometry $V\coloneqq(\mathbb{1}_{\mathrm{out}}\otimes(T_{\Omega}^{\tau})^{1/2})(|\mathbb{1}_{\mathrm{out}}\rangle\rangle\otimes\mathbb{1}_{\mathrm{in}})$ and ζ_{ω} is the POVM density $\zeta_{\omega}\coloneqq(T_{\Omega}^{-1/2}(\mathbb{1}_{\mathrm{out}}\otimes A_{\omega}^{*})S_0(\mathbb{1}_{\mathrm{out}}\otimes A_{\omega}^{\tau})T_{\Omega}^{-1/2})^{\tau}$.

Feed-forward schemes have recently attracted a remarkable interest in quantum optics and have been experimentally demonstrated in several applications, such as signal amplification,[9] coherent state cloning,[10] minimum-disturbance estimation,[11] and squeezed state purification.[12] In the finite-dimensional case, frame-orbit instruments provide the most general mathematical framework in which similar realization schemes can be searched.

V. GROUP-COVARIANT INSTRUMENTS WITH TRANSITIVE OUTCOME SPACE

A particular case of frame-orbit instruments is that of covariant instruments whose outcome space Ω is a transitive $\mathbf{G}$-space. Given a group $\mathbf{G}$, we will denote by $\mathsf{U}_{\mathbf{G}} := \{U_g \in \mathsf{Lin}(\mathsf{H}_{\text{in}})\}_{g\in\mathbf{G}}$, by $\mathsf{V}_{\mathbf{G}} := \{V_g \in \mathsf{Lin}(\mathsf{H}_{\text{out}})\}_{g\in\mathbf{G}}$ two unitary representations of $\mathbf{G}$, and by $\mathcal{U}_{\mathbf{G}} := \{\mathcal{U}_g \in \mathsf{CP}(\mathsf{H}_{\text{in}},\mathsf{H}_{\text{in}})\}_{g\in\mathbf{G}}$, $\mathcal{V}_{\mathbf{G}} := \{\mathcal{V}_g \in \mathsf{CP}(\mathsf{H}_{\text{out}},\mathsf{H}_{\text{out}})\}_{g\in\mathbf{G}}$ the corresponding sets of automorphisms $\mathcal{U}_g(\cdot) := U_g \cdot U_g^\dagger$, $\mathcal{V}_g(\cdot) := V_g \cdot V_g^\dagger$.

Definition 3: (Group-covariant instruments) *Given a topological group* $\mathbf{G}$ *acting on* Ω, *and two continuous unitary (generally projective) representations* $\mathsf{U}_{\mathbf{G}}$ *and* $\mathsf{V}_{\mathbf{G}}$ *on the Hilbert spaces* H_{in} *and* H_{out}, *respectively, we say that the instrument* $\mathcal{Z}:\Sigma_\Omega \to \mathsf{CP}(\mathsf{H}_{\text{in}},\mathsf{H}_{\text{out}})$ *is group covariant with respect to* $(\mathbf{G},\mathcal{U}_{\mathbf{G}},\mathcal{V}_{\mathbf{G}})$ *when one has*

$$\mathcal{Z}_B \circ \mathcal{U}_g(\rho) = \mathcal{V}_g \circ \mathcal{Z}_{g^{-1}(B)}(\rho) \quad \forall\, \rho \in \mathsf{S}(\mathsf{H}_{\text{in}}), \quad \forall\, B \in \Sigma_\Omega, \quad \forall\, g \in \mathbf{G}, \tag{55}$$

with $g^{-1}(B) := \{\omega \in \Omega \,|\, g\omega \in B\}$.

In the case of transitive group action on the outcome space Ω, for any point $\omega_0 \in \Omega$ one has $\Omega = \mathbf{G}\omega_0$, and the outcome space Ω can be identified with the space of left cosets $\Omega \equiv \mathbf{G}/\mathbf{G}_0$ with respect to the stability group $\mathbf{G}_0 := \{h \in \mathbf{G} \,|\, h\omega_0 = \omega_0\}$. Denote by π the projection map $\pi:\mathbf{G} \to \Omega$, $g \mapsto \pi(g) = g\omega_0$, and by μ the invariant measure on Ω given by $\mu(B) = \int_{\pi^{-1}(B)} dg$, where dg is the normalized Haar measure over $\mathbf{G}$. Under this hypothesis and notation, the following structure theorem holds.[22,23]

Theorem 2: *Let* $\mathbf{G}$ *be a compact group*, $\mathbf{G}_0$ *be a closed subgroup, and* $\mathcal{Z}:\Sigma_\Omega \to \mathsf{CP}(\mathsf{H}_{\text{in}},\mathsf{H}_{\text{out}})$ *be a covariant instrument with respect to* $(\mathbf{G},\mathcal{U}_{\mathbf{G}},\mathcal{V}_{\mathbf{G}})$ *with outcome space* $\Omega = \mathbf{G}/\mathbf{G}_0$. *Then* $\mathcal{Z}$ *admits a density with respect to* μ *of the form*

$$\mathcal{S}_{\pi(g)} = \mathcal{V}_g \mathcal{S}_0 \mathcal{U}_g^\dagger \quad \forall\, g \in \mathbf{G}, \tag{56}$$

where $\mathcal{S}_0$ *is a CP map,* $\mathcal{V}_g(\cdot) := V_g \cdot V_g^\dagger$, *and* $\mathcal{U}_g^\dagger(\cdot) := U_g^\dagger \cdot U_g$.

From now on we will confine our attention to compact groups $\mathbf{G}$. Since the group action is transitive, Eq. (56) defines the density $\mathcal{S}_\omega$ for any $\omega \in \Omega$. Indeed one can take any measurable section $\sigma:\Omega \to \mathbf{G}$, $\omega \mapsto \sigma(\omega)$, $\pi(\sigma(\omega)) = \omega$ and declare

$$\mathcal{S}_\omega = \mathcal{V}_{\sigma(\omega)} \mathcal{S}_0 \mathcal{U}_{\sigma(\omega)}^\dagger. \tag{57}$$

The above equation clearly characterizes any covariant instrument with $\Omega = \mathbf{G}/\mathbf{G}_0$ as a frame-orbit instrument with respect to $(\mu,\mathcal{U}_{\sigma(\Omega)},\mathcal{V}_{\sigma(\Omega)})$. In addition, Eq. (56) implies the invariance condition,

$$\mathcal{S}_0 = \mathcal{V}_h \mathcal{S}_0 \mathcal{U}_h^\dagger \quad \forall\, h \in \mathbf{G}_0, \tag{58}$$

which in terms of CJ operators becomes the commutation relation,

$$[S_0, V_h \otimes U_h^*] = 0 \quad h \in \mathbf{G}_0. \tag{59}$$

A. Covariant POVMs and dilation of covariant instruments

1. Minimal dilation

Let $\mathcal{Z}$ be an instrument with outcome space $\Omega = \mathbf{G}/\mathbf{G}_0$ covariant with respect to $(\mathbf{G},\mathcal{U}_{\mathbf{G}},\mathcal{V}_{\mathbf{G}})$. From Eqs. (50) and (57), the CJM density is

$$S_\omega = (\mathcal{V}_{\sigma(\omega)} \otimes \mathcal{U}^*_{\sigma(\omega)})(S_0). \tag{60}$$

Exploiting the Mackey–Bruhat identity, which sets up an isomorphism between $\mathbf{G}$ equipped with the Haar measure dg and $\Omega \times \mathbf{G}_0$ equipped with the product measure $\mu(d\omega)\times\nu(dh)$, $\nu(dh)$ normalized Haar measure over $\mathbf{G}_0$ (see, e.g., Ref. 24), we obtain

$$Z_\Omega = \int_\Omega \mu(d\omega) S_\omega = \int_{\Omega\times\mathbf{G}_0} \mu(d\omega)\nu(dh)(\mathcal{V}_{\sigma(\omega)} \otimes \mathcal{U}^*_{\sigma(\omega)})(S_0) = \int_{\mathbf{G}} dg(\mathcal{V}_g \otimes \mathcal{U}^*_g)(S_0). \tag{61}$$

As a consequence, we have $[Z_\Omega, V_g \otimes U^*_g]=0 \quad \forall g \in \mathbf{G}$, and the density ζ_ω in Eq. (51) is given by

$$\zeta_\omega = (V^*_{\sigma(\omega)} \otimes U_{\sigma(\omega)})\Xi(V^*_{\sigma(\omega)} \otimes U_{\sigma(\omega)})^\dagger \quad \Xi \coloneqq (Z_\Omega^{-1/2} S_0 Z_\Omega^{-1/2})^\tau, \tag{62}$$

with Ξ satisfying the commutation relation

$$[\Xi, V^*_h \otimes U_h] = 0 \quad \forall\, h \in \mathbf{G}_0. \tag{63}$$

This shows that POVM $Q_B = \int_B \mu(d\omega)\zeta_\omega$ used in the minimal dilation of Theorem 1 is a covariant POVM with outcome space Ω,[23] i.e.,

$$Q_{g(B)} = (\mathcal{V}^*_g \otimes \mathcal{U}_g)(Q_B) \quad \forall\, B \in \Sigma_\Omega. \tag{64}$$

Furthermore, using the relation $[Z^\tau_\Omega, V^*_g \otimes U_g] = [Z^*_\Omega, V^*_g \otimes U_g] = 0 \quad \forall g \in \mathbf{G}$, it is immediate to see that the minimal isometry $V = (\mathbb{1}_{\mathrm{out}} \otimes (Z^\tau_\Omega)^{1/2})(|\, \mathbb{1}_{\mathrm{out}}\rangle\rangle \otimes \mathbb{1}_{\mathrm{in}})$ intertwines the two representations $\mathsf{V}_{\mathbf{G}} \otimes \mathsf{V}^*_{\mathbf{G}} \otimes \mathsf{U}_{\mathbf{G}}$ and $\mathsf{U}_{\mathbf{G}}$, namely,

$$(V_g \otimes V^*_g \otimes U_g)V = VU_g \quad \forall\, g \in \mathbf{G}. \tag{65}$$

2. Minimal feed-forward realization and generalized teleportation schemes

A minimal feed-forward realization can be obtained by using Corollary 1, in terms of the instrument $\mathcal{T}_B \coloneqq \int_B \mu(d\omega)\mathcal{S}_0 \mathcal{U}^\dagger_{\sigma(\omega)}$. In this case we have

$$T_\Omega = \int_{\mathbf{G}} dg(\mathcal{I}_{\mathrm{out}} \otimes \mathcal{U}^*_g)(S_0), \tag{66}$$

whence $[T_\Omega, \mathbb{1}_{\mathrm{out}} \otimes U^*_g]=0 \quad \forall g \in \mathbf{G}$. As a consequence, the POVM density ζ_ω is now given by

$$\zeta_\omega = (I \otimes U_{\sigma(\omega)})\Xi(I \otimes U_{\sigma(\omega)})^\dagger \quad \Xi \coloneqq (T_\Omega^{-1/2} S_0 T_\Omega^{-1/2})^\tau. \tag{67}$$

Notice that in this case ζ_ω is not a covariant POVM density, since the relation $[\Xi, \mathbb{1}_{\mathrm{out}} \otimes U_h] = 0 \quad \forall h \in \mathbf{G}_0$ does not necessarily hold. The minimal isometry V is now given by $V = (\mathbb{1}_{\mathrm{out}} \otimes (T^\tau_\Omega)^{1/2})(|\, \mathbb{1}_{\mathrm{out}}\rangle\rangle \otimes \mathbb{1}_{\mathrm{in}})$ and enjoys the property

$$(\mathbb{1}^{\otimes 2}_{\mathrm{out}} \otimes U_g)V = VU_g \quad \forall\, g \in \mathbf{G}. \tag{68}$$

For irreducible $\mathsf{U}_{\mathbf{G}}$ the above equation yields $V = |\, \Psi\rangle\rangle \otimes \mathbb{1}_{\mathrm{in}}$ for some $|\, \Psi\rangle\rangle \in \mathsf{H}^{\otimes 2}_{\mathrm{out}}$, $\|\Psi\|_2 = 1$, namely, the isometry V is just the extension with some pure state. Precisely, computing the average in Eq. (66), we have $T_\Omega = d_{\mathrm{in}}^{-1} \operatorname{Tr}_{\mathrm{in}}[S_0] \otimes \mathbb{1}_{\mathrm{in}}$, whence

$$V = |\, \sigma^{1/2}\rangle\rangle \otimes \mathbb{1}_{\mathrm{in}} \quad \sigma \coloneqq d_{\mathrm{in}}^{-1} \operatorname{Tr}_{\mathrm{in}}[S_0] = d_{\mathrm{in}}^{-1} \sum_{i=1}^{r} S_i S_i^\dagger. \tag{69}$$

The feed-forward realization then becomes a generalized teleportation scheme where $|\, \sigma^{1/2}\rangle\rangle$ plays the role of entangled resource, the joint measurement ζ_ω is performed by the sender on the input system and on half of the entangled state, the outcome ω is classically transmitted, and the conditional operation $\mathcal{V}_{\sigma(\omega)}$ is performed at the receiver's end. The discussion on generalized teleportation schemes will be extended in Sec. VI to the case of frame-orbit instruments generated by tight operator frames.

3. Nonminimal feed-forward dilations

Using group theory it is easy to construct nonminimal dilations of group-covariant instruments. Let us decompose H_{in} and $\mathsf{U}_{\mathbf{G}}$ as

$$\mathsf{H}_{\mathrm{in}} = \bigoplus_{\mu\in\mathsf{S}} \mathsf{H}_\mu \otimes \mathbb{C}^{m_\mu}, \quad U_g = \bigoplus_{\mu\in\mathsf{S}} U_g^\mu \otimes 1_{m_\mu}, \tag{70}$$

the sum running over the set S of inequivalent irreducible representations (irreps) of $\mathbf{G}$ contained in the decomposition of $\mathsf{U}_{\mathbf{G}}$, $\mathsf{H}_\mu(\mathbb{C}^{m_\mu})$ being the representation (multiplicity) space of the irrep μ, of dimension $\dim(\mathsf{H}_\mu)=d_\mu$ ($\dim \mathbb{C}^{m_\mu}=m_\mu$). The group average of an operator $O\in\mathsf{Lin}(\mathsf{H}_{\mathrm{in}})$ is then given by

$$\langle O\rangle_{\mathsf{U}_{\mathbf{G}}} \coloneqq \int_{\mathbf{G}} dg U_g O U_g^\dagger = \bigoplus_{\mu\in\mathsf{S}} d_\mu^{-1}(1_{d_\mu}\otimes \mathrm{Tr}_{\mathsf{H}_\mu}[\Pi_\mu O \Pi_\mu]), \tag{71}$$

Π_μ denoting the projector onto $\mathsf{H}_\mu\otimes\mathbb{C}^{m_\mu}$. For the dilation we introduce now two ancillary spaces $\mathsf{H}_0\simeq\mathbb{C}^r$, where $r=\mathrm{rank}(S_0)$, and $\widetilde{\mathsf{H}}$, which is given by

$$\widetilde{\mathsf{H}} \coloneqq \bigoplus_{\mu\in\mathsf{S}} \mathsf{H}_\mu\otimes\mathbb{C}^{d_\mu}, \tag{72}$$

and carries the representation $\widetilde{\mathsf{U}}_{\mathbf{G}}\coloneqq\{\widetilde{U}_g=\oplus_{\mu\in\mathsf{S}}U_g^\mu\otimes 1_{d_\mu}\}$.

Proposition 3: *Let $\mathcal{Z}$ be an instrument with outcome space $\Omega=\mathbf{G}/\mathbf{G}_0$, covariant with respect to $\{\mathbf{G},\mathcal{U}_{\mathbf{G}},\mathcal{V}_{\mathbf{G}}\}$, and with density $\mathcal{S}_\omega$. A dilation of $\mathcal{Z}$ can be achieved as follows:*

$$\mathcal{S}_{\pi(g)}(\rho) = V_g(\mathrm{Tr}_{\mathsf{H}_0}\mathrm{Tr}_{\widetilde{\mathsf{H}}}[(1_{\mathsf{H}_0}\otimes 1_{\mathrm{out}}\otimes\zeta'_g)V'\rho V'^\dagger])V_g^\dagger, \tag{73}$$

where $\pi:g\mapsto\pi(g)\in\mathbf{G}/\mathbf{G}_0$ projects group elements to the corresponding left coset, ζ'_g is the POVM density on $\widetilde{\mathsf{H}}$ given by

$$\zeta'_g = |\eta_g\rangle\langle\eta_g| \quad |\eta_g\rangle = \bigoplus_{\mu\in\mathsf{S}} \sqrt{d_\mu}|U_g^\mu\rangle\rangle \in \widetilde{\mathsf{H}}, \tag{74}$$

and $V':\mathsf{H}_{\mathrm{in}}\to\mathsf{H}_0\otimes\mathsf{H}_{\mathrm{out}}\otimes\widetilde{\mathsf{H}}$ is the isometry

$$V' = \sum_{i=1}^r |i\rangle\otimes\int_{\mathbf{G}} dg S_i U_g^\dagger\otimes|\eta_g\rangle, \tag{75}$$

$\{|i\rangle\}_{i=1}^r$ being an orthonormal basis for H_0.

Proof: As an immediate consequence of Eq. (71), the vectors $|\eta_g\rangle=\widetilde{U}_g|\eta\rangle$ provide a resolution of the identity in $\widetilde{\mathsf{H}}$, namely,

$$\int_{\mathbf{G}} dg|\eta_g\rangle\langle\eta_g| = 1_{\widetilde{\mathsf{H}}}, \tag{76}$$

whence ζ'_g is the density of a normalized POVM. Moreover, it is easy verify that V' is an isometry. First, we have $\langle\eta_g|\eta_h\rangle=\Sigma_{\mu\in\mathsf{S}}d_\mu\chi_\mu(g^{-1}h)$, where $\chi_\mu(g)\doteq\mathrm{Tr}[U_g^\mu]$ is the character of the irrep μ. Then, as a consequence of the orthogonality of irreducible matrix elements, we have the relation

$$\int_{\mathbf{G}} dg\left(\sum_{\mu} d_\mu \chi_\mu^*(g)\right) U_g = \mathbb{1}_{\mathsf{in}}, \tag{77}$$

whence

$$\begin{aligned} V'^\dagger V' &= \sum_i \int_{\mathbf{G}} dg \int_{\mathbf{G}} dh U_g S_i^\dagger S_i U_h^\dagger \langle \eta_g | \eta_h \rangle = \int_{\mathbf{G}} dg \int_{\mathbf{G}} dh U_g \xi U_h^\dagger \left(\sum_{\mu} d_\mu \chi_\mu(g^{-1}h)\right) \\ &= \int_{\mathbf{G}} dg \int_{\mathbf{G}} dk U_g \xi U_k U_g^\dagger \left(\sum_{\mu} d_\mu \chi_\mu^*(k)\right) = \int_{\mathbf{G}} dg U_g \xi U_g^\dagger = \mathbb{1}_{\mathsf{in}}, \end{aligned} \tag{78}$$

having used Eq. (46) with $\mathsf{A}_\Omega \equiv \mathsf{U}_{\mathbf{G}}$. Finally, identity (73) holds, namely,

$$\begin{aligned} V_g(\mathrm{Tr}_{\mathsf{H}_0}\mathrm{Tr}_{\tilde{\mathsf{H}}}[V'\rho V'^\dagger(\mathbb{1}_A \otimes \mathbb{1}_{\mathsf{H}} \otimes \zeta_g')])V_g^\dagger &= V_g\left(\sum_{i=1}^r \int_{\mathbf{G}} dh \int_{\mathbf{G}} dk S_i U_h^\dagger \rho U_k S_i^\dagger \langle \eta_g | \eta_h \rangle \langle \eta_k | \eta_g \rangle\right) V_g^\dagger \\ &= V_g\left(\sum_{i=1}^r \int_{\mathbf{G}} dh \int_{\mathbf{G}} dk S_i U_h U_g^\dagger \rho U_g U_k S_i^\dagger \left(\sum_{\mu \in \mathsf{S}} d_\mu \chi_\mu^*(h)\right)\right. \\ &\quad \left.\times \left(\sum_{\nu \in \mathsf{S}} d_\nu \chi_\nu^*(k)\right)\right) V_g^\dagger = V_g\left(\sum_{i=1}^r S_i U_g^\dagger \rho U_g S_i^\dagger\right) V_g^\dagger \\ &= \mathcal{V}_g \mathcal{S}_0 \mathcal{U}_g^\dagger(\rho) = \mathcal{S}_{\pi(g)}(\rho), \end{aligned} \tag{79}$$

having used Eqs. (77) and (56). □

The above proposition shows that in order to realize the instrument $\mathcal{Z}$, it is enough to perform the indirect measurement ζ_g—whose outcome space is the whole group $\mathbf{G}$—and subsequently to use the classical data processing $g \mapsto \pi(g)$ that projects the g onto the final outcome space $\Omega = \mathbf{G}/\mathbf{G}_0$. In this way, both the statistics and the state reduction associated with the operational scheme of measurement and feed-forward are exactly the same as for the instrument $\mathcal{Z}$.

4. Naimark dilation

Consider the Hilbert space $\oplus_{\mu \in \hat{\mathbf{G}}} \mathsf{H}_\mu \otimes \mathbb{C}^{d_\mu}$, where $\hat{\mathbf{G}}$ denotes the set of all possible unitary irreps of $\mathbf{G}$. According to Fourier–Plancherel theory,[24] any vector $|f\rangle \in \oplus_{\mu \in \hat{\mathbf{G}}} \mathsf{H}_\mu \otimes \mathbb{C}^{d_\mu}$ is associated with a square-summable function $f(g)$ as follows:

$$|f\rangle \mapsto f(g) = \sum_{\mu \in \hat{\mathbf{G}}} \sqrt{d_\mu} \langle\langle U_g \,| \Pi_\mu | f\rangle. \tag{80}$$

In this way, one has $\langle f | h \rangle = \int_{\mathbf{G}} dg f^*(g) h(g)$, and correspondence $|f\rangle \mapsto f(g)$ sets up a unitary equivalence between the Hilbert spaces $\oplus_{\mu \in \hat{\mathbf{G}}} \mathsf{H}_\mu \otimes \mathbb{C}^{d_\mu}$ and $L^2(\mathbf{G}, dg)$. Therefore, we can identify the ancilla space $\tilde{\mathsf{H}}$ in Proposition 3 with a subspace of $L^2(\mathbf{G}, dg)$, the projector on $\tilde{\mathsf{H}}$ being $Y = \oplus_{\mu \in \mathsf{S}} \Pi_\mu$. Hence the POVM Q' defined by the density ζ_g' in Eq. (74) has the following Naimark dilation:

$$\begin{aligned} \langle f | Q_B' | h \rangle &= \int_B dg \sum_{\mu, \nu \in \mathsf{S}} \sqrt{d_\mu d_\nu} \langle f | \Pi_\mu | \; U_g^\mu \rangle\rangle \langle\langle U_g^\nu \; | \Pi_\nu | h \rangle = \int_B dg (Yf)^*(g)(Yh)(g) \\ &= \langle f | Y^\dagger E_B Y | h \rangle \quad \forall \, |f\rangle, |h\rangle \in L^2(\mathbf{G}, dg), \end{aligned} \tag{81}$$

where E is the PVM on $L^2(\mathbf{G}, dg)$ defined by

$$\langle f|E_B|h\rangle := \int_B dg f^*(g)h(g) \quad \forall |f\rangle,|h\rangle \in L^2(\mathbf{G},dg). \tag{82}$$

The relation $Q_B = Y^\dagger E_B Y$ shows that the POVM Q' is simply the projection of the PVM E on the subspace $\tilde{\mathsf{H}} \subset L^2(\mathbf{G},dg)$. It is worth noting that the POVM Q' is also the optimal POVM for the estimation of an unknown unitary transformation $\tilde{U}_g$ acting on the finite-dimensional Hilbert space $\tilde{\mathsf{H}}$.[25]

VI. TIGHT OPERATOR FRAMES: TELEINSTRUMENTS AND TELECHANNELS

Let μ be a finite measure on Ω and A_Ω a measurable family of operators. A_Ω is a tight operator frame if the frame operator A is the identity on $\mathsf{H}^{\otimes 2}$, i.e.,

$$A = \int_\Omega \mu(d\omega) | A_\omega\rangle\rangle\langle\langle A_\omega | = \mathbb{1} \otimes \mathbb{1}. \tag{83}$$

A special case of tight unitary frame is that of irreducible unitary representation of a compact group $\mathbf{G}$, namely, $\Omega = \mathbf{G}$, $\mathsf{A}_\Omega = \mathsf{U}_{\mathbf{G}}$, and $\mu(d\omega) = \dim \mathsf{H} \times dg$.

Generalizing the notion of tight frame to the case where the frame operator is the identity only on the first copy, we have the following.

Definition 4: *Let μ be a measure on Ω and A_Ω be a measurable family of operators. We say that A_Ω is a* left-tight operator frame *if*

$$\int_\Omega \mu(d\omega) | A_\omega\rangle\rangle\langle\langle A_\omega | = \mathbb{1} \otimes K \tag{84}$$

for some positive operator $0 \le K \in \mathsf{Lin}(\mathsf{H})$.

Note that identity (84) is equivalent to the following ones:

$$\int_\Omega \mu(d\omega) A_\omega X A_\omega^\dagger = \mathrm{Tr}[X K^\tau]\, \mathbb{1}, \quad \forall X \in \mathsf{Lin}(\mathsf{H}),$$

$$\int_\Omega \mu(d\omega) A_\omega \otimes A_\omega^* = | \mathbb{1}\rangle\rangle\langle\langle K^\tau |. \tag{85}$$

We will see in the following that operator-frame instruments generated by operations $\mathcal{A}_\omega(\cdot) = A_\omega \cdot A_\omega^\dagger$ corresponding to left-tight frames A_Ω can be realized by generalized teleportation schemes, in which two parties (conventionally named Alice and Bob) exploit an entangled resource to achieve the instrument via local operations and one-way classical communication: a suitable joint POVM ζ_ω is measured by Alice on the input, and on one side of the entangled state, the measurement outcome ω is announced to Bob, who performs a conditional feed-forward operation $\mathcal{B}_\omega$ on the other side. We will also use the term teleinstruments to denote instruments that admit such a realization. In addition, we will show that frame-orbit instruments generated by tight unitary frames are useful for the realization of covariant channels. In this case, which covers, in particular, the case of unitary irreducible group representations, a covariant channel can be realized by a generalized teleportation scheme, hence becoming a telechannel. In particular, we will provide also the realization of covariant channels such as universal cloning[26] and universal NOT.[27]

A. Minimal teleinstruments

Let $\mathcal{Z}:\Sigma_\Omega \to \mathsf{CP}(\mathsf{H}_{\mathrm{in}},\mathsf{H}_{\mathrm{out}})$ be a frame-orbit instrument with respect to $(\mu,\mathcal{A}_\Omega,\mathcal{B}_\Omega)$ with $\mathcal{A}_\Omega$ left-tight operator frame, and let $\mathcal{S}_\omega$ be the instrument density of $\mathcal{Z}$. We consider now the minimal feed-forward realization of Corollary 1. Using Eq. (85), the CJ operator of the instrument channel $\mathcal{T}_\Omega$ is given by

$$T_\Omega = (\mathcal{T}_\Omega \otimes \mathcal{I})(|\,\mathbb{1}_{\mathrm{in}}\rangle\rangle\langle\langle \mathbb{1}_{\mathrm{in}}\,|) = \int_\Omega \mu(d\omega)(\mathbb{1}_{\mathrm{out}} \otimes A^*_\omega)S_0(\mathbb{1}_{\mathrm{out}} \otimes A^\tau_\omega) = \mathrm{Tr}_{\mathrm{in}}[S_0(\mathbb{1}_{\mathrm{out}} \otimes (K^*)^\tau)] \otimes \mathbb{1}_{\mathrm{in}}$$

$$= \mathrm{Tr}_{\mathrm{in}}[S_0(\mathbb{1}_{\mathrm{out}} \otimes K)] \otimes \mathbb{1}_{\mathrm{in}} = \sigma \otimes \mathbb{1}_{\mathrm{in}}, \tag{86}$$

having used the fact that $(K^*)^\tau = K$ since $K \geq 0$, and having defined the state

$$\sigma := \mathrm{Tr}_{\mathrm{in}}[S_0(\mathbb{1}_{\mathrm{out}} \otimes K)] = \mathcal{S}_0(K^\tau) \quad \mathrm{Tr}[\sigma] = 1. \tag{87}$$

The minimal isometry $V = (\mathbb{1}_{\mathrm{out}} \otimes T_\Omega^{\tau 1/2})|\,\mathbb{1}_{\mathrm{out}}\rangle\rangle \otimes \mathbb{1}_{\mathrm{in}}$ is then given by

$$V = |\,\sigma^{1/2}\rangle\rangle \otimes \mathbb{1}_{\mathrm{in}} \tag{88}$$

and Corollary 1 yields the following generalized teleportation scheme for the instrument $\mathcal{Z}$:

$$\mathcal{S}_\omega(\rho) = \mathcal{B}_\omega(\mathrm{Tr}_{2,3}[(\,|\sigma^{1/2}\rangle\rangle\langle\langle\sigma^{1/2}\,| \otimes \rho)(\mathbb{1}_{\mathrm{out}} \otimes \zeta_\omega)]), \tag{89}$$

where the POVM density ζ_ω is given by $\zeta_\omega = (\sigma^{-1/2\tau} \otimes A_\omega)S_0^\tau(\sigma^{-1/2\tau} \otimes A^\dagger_\omega)$.

In conclusion, in the minimal feed-forward realization the frame-orbit instrument $\mathcal{Z}$ can be implemented by two parties that share the pure entangled state $|\sigma^{1/2}\rangle\rangle$ by using only local operations and one-way classical communication: it is enough for Alice to perform the joint POVM $\zeta_\omega = (\sigma^{-1/2\tau} \otimes A_\omega)S_0^\tau(\sigma^{-1/2\tau} \otimes A^\dagger_\omega)$ on the input state and on one side of the entangled resource and to announce the measurement outcome ω to Bob, who implements the conditional channel $\mathcal{B}_\omega$.

B. Nonminimal teleinstruments

Starting from the minimal dilation, it is simple to obtain other generalized teleportation realization schemes. In particular, from Eq. (89) we obtain

$$\begin{aligned}
\mathcal{S}_\omega &= \mathcal{B}_\omega(\mathrm{Tr}_{2,3}[(\,|\sigma^{1/2}\rangle\rangle\langle\langle\sigma^{1/2}\,| \otimes \rho)(\mathbb{1}_{\mathrm{out}} \otimes (\sigma^{-1/2\tau} \otimes A_\omega)S_0^\tau(\sigma^{-1/2\tau} \otimes A^\dagger_\omega))]) \\
&= \mathcal{B}_\omega(\mathrm{Tr}_{2,3}[(\,|\mathbb{1}_{\mathrm{out}}\rangle\rangle\langle\langle\mathbb{1}_{\mathrm{out}}\,| \otimes \rho)(\mathbb{1}_{\mathrm{out}} \otimes (\mathcal{S}_0^\tau \otimes \mathcal{I})(\,|A^\tau_\omega\rangle\rangle\langle\langle A^\tau_\omega\,|))]) \\
&= \mathcal{B}_\omega(\mathrm{Tr}_{2,3}[((\mathcal{I} \otimes \mathcal{S}_0^\tau)(\,|\mathbb{1}_{\mathrm{out}}\rangle\rangle\langle\langle\mathbb{1}_{\mathrm{out}}\,|) \otimes \rho)(\mathbb{1}_{\mathrm{out}} \otimes \,|A^\tau_\omega\rangle\rangle\langle\langle A^\tau_\omega\,|)]) \\
&= \mathcal{B}_\omega(\mathrm{Tr}_{2,3}[((\mathcal{S}_0 \otimes \mathcal{I})(\,|\mathbb{1}_{\mathrm{out}}\rangle\rangle\langle\langle\mathbb{1}_{\mathrm{out}}\,|) \otimes \rho)(\mathbb{1}_{\mathrm{out}} \otimes \,|A^\tau_\omega\rangle\rangle\langle\langle A^\tau_\omega\,|)]) \\
&= \mathcal{B}_\omega(\mathrm{Tr}_{2,3}[((\mathcal{S}_0 \otimes \mathcal{I})(\,|K^{\tau 1/2}\rangle\rangle\langle\langle K^{\tau 1/2}\,|) \otimes \rho)(\mathbb{1}_{\mathrm{out}} \otimes \,|K^{-1/2}A^\tau_\omega\rangle\rangle\langle\langle K^{-1/2}A^\tau_\omega\,|)]) \\
&= \mathcal{B}_\omega(\mathrm{Tr}_{2,3}[(\sigma' \otimes \rho)(\mathbb{1}_{\mathrm{out}} \otimes \zeta'_\omega)])
\end{aligned} \tag{90}$$

having defined the state

$$\sigma' := (\mathcal{S}_0 \otimes \mathcal{I})(|\,K^{\tau 1/2}\rangle\rangle\langle\langle K^{\tau 1/2}\,|) \tag{91}$$

and the POVM density

$$\zeta_\omega := |\,K^{-1/2}A^\tau_\omega\rangle\rangle\langle\langle K^{-1/2}A^\tau_\omega\,|. \tag{92}$$

The normalization of σ' is given by $\mathrm{Tr}[\sigma'] = \mathrm{Tr}[\mathcal{S}_0(K^\tau)] = \mathrm{Tr}[\sigma] = 1$ [having used Eq. (87)], while the normalization of the POVM ζ'_ω follows directly from Eq. (84).

Also the above feed-forward realization is a generalized teleportation scheme, which allows Alice and Bob to implement the instrument $\mathcal{Z}$ as a teleinstrument. Notice that now the entangled resource $\sigma' = (\mathcal{S}_0 \otimes \mathcal{I})(|K^{\tau 1/2}\rangle\rangle\langle\langle K^{\tau 1/2}|)$ is a generally *mixed* state, whereas the joint POVM performed by Alice has now the *rank-one* density $\zeta'_\omega = |K^{-1/2}A_\omega^\tau\rangle\rangle\langle\langle K^{-1/2}A_\omega^\tau|$.

C. Tight unitary frames and Bell measurements

A particularly interesting generalized teleportation scheme arises when the left-tight frame A_Ω consists of unitary operators, namely, $\mathsf{A}_\Omega \equiv \mathsf{U}_\Omega$. This is the case, for instance, when A_ω is a unitary irreducible representation of some compact group $\mathbf{G}$.

It is immediate to see that a unitary left-tight frame A_Ω is necessarily tight, since Eq. (84) implies $K = \mathbb{1}_{\mathsf{in}}/d_{\mathsf{in}}$. For unitary tight frames, the nonminimal realization of Eq. (90) becomes

$$\mathcal{S}_\omega = \mathcal{B}_\omega\left(\mathrm{Tr}_{2,3}\left[\left(\frac{S_0}{d_{\mathsf{in}}} \otimes \rho\right)(\mathbb{1}_{\mathsf{out}} \otimes \zeta'_\omega)\right]\right) \; \zeta'_\omega = d_{\mathsf{in}}|U_\omega^\tau\rangle\rangle\langle\langle U_\omega^\tau|, \tag{93}$$

and the joint POVM used by Alice is a Bell measurement, i.e., ζ'_ω are proportional to rank-one projectors on maximally entangled states. On the other hand, the minimal realization of Eq. (89) gives

$$\mathcal{S}_\omega(\rho) = \mathcal{B}_\omega(\mathrm{Tr}_{2,3}[(|\sigma^{1/2}\rangle\rangle\langle\langle\sigma^{1/2}| \otimes \rho)(\mathbb{1}_{\mathsf{out}} \otimes \zeta_\omega)]), \tag{94}$$

with $\sigma = \mathcal{S}_0(\mathbb{1}_{\mathsf{in}})/d_{\mathsf{in}}$ and POVM density $\zeta_\omega = (\sigma^{-1/2\tau} \otimes U_\omega) S_0^\tau (\sigma^{-1/2\tau} \otimes U_\omega^\dagger)$. Note that typically the POVM ζ_ω in the minimal realization is not a Bell POVM.

VII. REALIZATION OF COVARIANT CHANNELS

The realization of a covariant instrument $\mathcal{Z}$ also allows one to achieve the corresponding channel $\mathcal{Z}_\Omega$ by simply averaging over the instrument outcomes. Therefore, the general realization schemes presented in Theorem 1 and Corollary 1 for instruments can be directly transferred to the corresponding channels. In particular, for any teleinstrument $\mathcal{Z}$ in Secs. VI B and VI A, we have a corresponding telechannel $\mathcal{Z}_\Omega$ achieved by the same generalized teleportation scheme.

A particularly interesting case is that of covariant channels, which we intend here in a very broad sense, according to the following.

Definition 5: *Let $\mathcal{A}_\Omega$ be a family of quantum channels on H_{in} and $\mathcal{B}_\Omega$ a family of quantum channels on $\mathsf{H}_{\mathsf{out}}$. A channel $\mathcal{C} \in \mathsf{CP}(\mathsf{H}_{\mathsf{in}}, \mathsf{H}_{\mathsf{out}})$ is covariant with respect to $(\mathcal{A}_\Omega, \mathcal{B}_\Omega)$ if*

$$\mathcal{C}\mathcal{A}_\omega = \mathcal{B}_\omega\mathcal{C} \quad \forall\, \omega \in \Omega. \tag{95}$$

In particular, we consider the case where all channels in $\mathcal{A}_\Omega$ are unitary, namely, $\mathcal{A}_\omega(\cdot) \equiv \mathcal{U}_\omega(\cdot) = U_\omega \cdot U_\omega^\dagger$, for some unitary operator $U_\omega \in \mathsf{Lin}(\mathsf{H}_{\mathsf{in}})$. Since for a covariant channel one has $\mathcal{C} = \mathcal{B}_\omega\mathcal{C}\mathcal{U}_\omega^\dagger \;\; \forall\, \omega \in \Omega$, any covariant channel is trivially the channel corresponding to a frame-orbit instrument, namely, $\mathcal{C} \equiv \mathcal{Z}_\Omega$ with $\mathcal{Z}_B := \int_B \mu(d\omega)\mathcal{S}_\omega$, $\mathcal{S}_\omega := \mathcal{B}_\omega\mathcal{C}\mathcal{U}_\omega^\dagger$. In fact, the covariant channel $\mathcal{C}$ coincides with the instrument density $\mathcal{S}_\omega$ for any outcome ω. In particular, when U_Ω is a tight unitary frame, $\mathcal{C}$ becomes a telechannel, and the nonminimal dilation of Eq. (93) yields a generalized teleportation scheme with Bell measurement,

$$\mathcal{C}(\rho) \equiv \mathcal{S}_\omega(\rho) = \mathcal{B}_\omega(\mathrm{Tr}_{2,3}[(C/d_{\mathsf{in}} \otimes \rho)(\mathbb{1}_{\mathsf{out}} \otimes d_{\mathsf{in}}|U_\omega^\tau\rangle\rangle\langle\langle U_\omega^\tau|)]), \tag{96}$$

with $C = (\mathcal{C} \otimes \mathcal{I})(|\mathbb{1}_{\mathsf{in}}\rangle\rangle\langle\langle\mathbb{1}_{\mathsf{in}}|) \equiv (\mathcal{S}_0 \otimes \mathcal{I})(|\mathbb{1}_{\mathsf{in}}\rangle\rangle\langle\langle\mathbb{1}_{\mathsf{in}}|) = S_0$.

The minimal dilation of Eq. (89) gives instead

$$\mathcal{C}(\rho) \equiv \mathcal{S}_\omega(\rho) = \mathcal{B}_\omega(\mathrm{Tr}_{2,3}[(|\sigma^{1/2}\rangle\rangle\langle\langle\sigma^{1/2}| \otimes \rho)(\mathbb{1}_{\mathsf{out}} \otimes \zeta_\omega)]), \tag{97}$$

where $\sigma = \mathcal{C}(\mathbb{1}_{\mathsf{in}})/d_{\mathsf{in}}$ according to Eq. (87), and POVM density given by $\zeta_\omega = (\sigma^{-1/2\tau} \otimes U_\omega) C^\tau (\sigma^{-1/2\tau} \otimes U_\omega^\dagger)$.

Notice that the nonminimal realization uses the CJ state $\sigma_{\mathrm{CJ}}=C/d_{\mathrm{in}}$ as entangled resource, while the minimal realization uses a purification of the local state $\sigma=\mathrm{Tr}_{\mathrm{in}}[\sigma_{\mathrm{CJ}}]$. The two realizations coincide (up to local unitaries on H_{in}) when the CJ operator C is rank-one, corresponding to unitary channels.

We conclude with the following examples of application.

A. Ideal teleportation

Ideal teleportation from Alice's to Bob's site is described by the identity channel $\mathcal{C}=\mathcal{I}$, which is a covariant channel with respect to $(\mathcal{U}_\Omega,\mathcal{U}_\Omega)$ for *any* unitary frame U_Ω, since trivially $\mathcal{C}\mathcal{U}_\omega=\mathcal{U}_\omega\mathcal{C}\ \ \forall\,\omega\in\Omega$. For tight unitary frames, Eqs. (97) and (96) coincide and give the realization

$$\mathcal{C}(\rho)=\mathcal{U}_\omega\left(\mathrm{Tr}_{2,3}\left[\left(\frac{|\,\mathbb{1}\rangle\rangle\langle\langle\mathbb{1}\,|}{d}\otimes\rho\right)(\mathbb{1}_{\mathrm{out}}\otimes d|\ U_\omega^\tau\rangle\rangle\langle\langle U_\omega^\tau\ |)\right]\right). \tag{98}$$

In other words, our general scheme retrieves all possible schemes to achieve ideal teleportation with Bell observables,[28,29] and, more generally, with Bell POVMs.[30]

B. Universal telecloning

The optimal quantum cloning of pure states from N input copies to M output copies is given by a channel $\mathcal{C}_{N,M}\in\mathsf{CP}(\mathsf{H}_\mathsf{N}^+,\mathsf{H}_\mathsf{M}^+)$, where H_k^+ denotes the totally symmetric subspace of the tensor product $\mathsf{H}^{\otimes k}$. The channel is covariant with respect to the irreducible representations $(\mathcal{U}_\mathbf{G}^{\otimes N},\mathcal{U}_\mathbf{G}^{\otimes M})$ of the group $\mathbf{G}=\mathbb{SU}(d)$, namely, $\mathcal{C}_{N,M}\mathcal{U}_g^{\otimes N}=\mathcal{U}_g^{\otimes M}\mathcal{C}_{N,M}\ \ \forall\,g\in\mathbb{SU}(d)$, and is given by[26,31]

$$\mathcal{C}_{N,M}(\rho)=\frac{d_N^+}{d_M^+}P_M^+(\rho\otimes\mathbb{1}^{\otimes(M-N)})P_M^+\quad \rho\in\mathsf{S}(\mathsf{H}_\mathsf{N}^+), \tag{99}$$

where $d_k^+:=\dim\mathsf{H}_k^+$ and P_k^+ is the projector on H_k^+. The realization of Eq. (96) then yields a generalized teleportation scheme with covariant Bell POVM,

$$\mathcal{C}_{N,M}(\rho)=\mathcal{U}_g^{\otimes M}\left(\mathrm{Tr}_{2,3}\left[\left(\frac{C_{N,M}}{d_N^+}\otimes\rho\right)(\mathbb{1}_{\mathrm{out}}\otimes d_N^+|\ U_g^{\tau\otimes N}\rangle\rangle\langle\langle U_g^{\tau\otimes N}\ |)\right]\right), \tag{100}$$

with $C_{N,M}=(\mathcal{C}_{N,M}\otimes\mathcal{I})(|\ \mathbb{1}_{\mathsf{H}_N^+}\rangle\rangle\langle\langle\mathbb{1}_{\mathsf{H}_N^+}\ |)$. On the other hand, the feed-forward scheme of Eq. (97) gives

$$\mathcal{C}_{N,M}(\rho)=\mathcal{U}_g^{\otimes M}\left(\mathrm{Tr}_{2,3}\left[\left(\frac{|\ P_M^+\rangle\rangle\langle\langle P_M^+\ |}{d_M^+}\otimes\rho\right)(\mathbb{1}_{\mathrm{out}}\otimes\zeta_g)\right]\right), \tag{101}$$

where ζ_g is the covariant POVM given by

$$\zeta_g=d_M^+(\mathbb{1}_{\mathrm{out}}\otimes U_g^{\otimes N})C_{N,M}^\tau(\mathbb{1}_{\mathrm{out}}\otimes U_g^{\dagger\otimes N}). \tag{102}$$

C. Optimal universal NOT gate

The optimal universal NOT is the channel from H_N^+ to H_M^+ with $\mathsf{H}=\mathsf{Span}\{|0\rangle,|1\rangle\}\simeq\mathbb{C}^2$ which transforms N copies of a pure state into one approximate copy of its orthogonal complement. The channel $\mathcal{N}$ is given by the measure-and-reprepare scheme,[32]

$$\mathcal{N}(\rho)=\int_{\mathbb{SU}(d)}dg\ \mathrm{Tr}[\rho\zeta_g]U_g|1\rangle\langle1|U_g^\dagger, \tag{103}$$

where ζ_g is the covariant POVM $\zeta_g=d_N^+(U_g|0\rangle\langle0|U_g^\dagger)^{\otimes N}$. By definition, $\mathcal{N}\equiv\mathcal{Z}_\Omega$, where $\mathcal{Z}$ the covariant channel with density $\mathcal{S}_g(\rho)=\mathrm{Tr}[\rho\zeta_g]U_g|1\rangle\langle1|U_g^\dagger=\mathcal{U}_g\mathcal{S}_0\mathcal{U}_g^{\dagger\otimes N}$, $\mathcal{S}_0(\rho)=\mathrm{Tr}[\rho(|0\rangle\times\langle0|)^{\otimes N}]|1\rangle\langle1|$. In this case, it easy to see that the minimal generalized teleportation scheme given by Eq. (89) coincides with the definition of the channel: indeed $\mathcal{N}$ is of the measure-and-reprepare form, and by definition it can be achieved via a measurement at Alice's site combined with a conditional state preparation at Bob's site. On the other hand, the nonminimal scheme of Eq. (90) gives

$$\mathcal{N}(\rho) = \int_{\mathrm{SU}(d)} dg \mathcal{U}_g (\mathrm{Tr}_{2,3}[(|1\rangle\langle 1| \otimes |0\rangle\langle 0|^{\otimes N} \otimes \rho)(\mathbb{1}_{\mathrm{out}} \otimes d_N^{+} | U_g^{\tau \otimes N}\rangle\rangle\langle\langle U_g^{\tau\otimes N} |)]). \qquad (104)$$

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