QUANTUM CLONING OPTIMAL FOR JOINT MEASUREMENTS

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Abstract　　We show that universally covariant cloning is not optimal for achieving joint measurements of noncommuting observables with minimum added noise. For such a purpose a cloning transformation that is covariant with respect to a restricted transformation group is needed.

Introduction

Perfect cloning of unknown quantum systems is forbidden by the laws of quantum mechanics [1]. However, a universal optimal cloning has been proposed [2], which has been proved to be optimal in terms of fidelity [3, 4]. A obvious relevant application of such optimal cloning is eavesdropping in quantum cryptography [5]. However, quantum cloning can be of practical interest as a tool to engineer novel schemes of quantum measurements, in particular for joint measurements of noncommuting observables. Quite unexpectedly, as we will show in the following, the universally covariant cloning is not ideal for this purpose. Here, instead, cloning must be optimized for a reduced covariance group, depending on the desired joint measurement, in order to make the measurement over the cloned copies perfectly equivalent to an optimal joint measurement over the original.

In the following we consider: i) the case of spin 1/2, and the use of the $1 \to 3$ universally covariant cloning to achieve a joint measurement of the spin components; ii) the case of harmonic oscillator, along with a $1 \to 2$ cloning transformation that is *not* universally covariant. We show that the POVM obtained in the first case does not lead to a minimum-added-noise measurement. On the contrary, the second way allows one to achieve the ideal joint measurement of conjugated variables.

Universally covariant cloning and joint measurement of spin components

The $1 \to 3$ cloning map is given by [4]

$$\rho_3 = \frac{1}{2} S_3 \left(|\psi\rangle\langle\psi| \otimes \mathbf{1}^{\otimes 2} \right) S_3 , \tag{1}$$

where $|\psi\rangle$ denotes the input state to be cloned, and S_3 is the projector on the space spanned by the vectors $\{|s_i\rangle\langle s_i|, i = 0 \div 3\}$, with $|s_0\rangle = |000\rangle$, $|s_1\rangle = 1/\sqrt{3}(|001\rangle + |010\rangle + |100\rangle)$, $|s_2\rangle = 1/\sqrt{3}(|011\rangle + |101\rangle + |110\rangle)$ and $|s_3\rangle = |111\rangle$ ($\{|0\rangle, |1\rangle\}$ being a basis for each spin 1/2 system). Independent measurements on the three copies along orthogonal axes provides the following POVM

$$\Pi(\vec{m}) = \frac{1}{2} \mathrm{Tr}_{2,3} \left[S_3 \, \Omega(\vec{m}) \, S_3 \right] . \tag{2}$$

where

$$\Omega(\vec{m}) = \frac{1}{8} (\mathbf{1} + m_x \sigma_x) \otimes (\mathbf{1} + m_y \sigma_y) \otimes (\mathbf{1} + m_z \sigma_z) , \tag{3}$$

is the product of projectors on the output copies in terms of Pauli matrices σ_α, and $m_\alpha = \pm 1$ corresponds to each of the outcomes. Explicit calculation gives

$$\Pi(\vec{m}) = \frac{1}{8} \left[\mathbf{1} + \frac{5}{9} \vec{m} \cdot \vec{\sigma} \right] . \tag{4}$$

where the factor $5/9$ comes from the shrinking of the Bloch vector due to the cloning map (1). In order to have the measurement unbiased, the spin component outcomes $m_\alpha = \pm 1$ must be rescaled to $m_\alpha = \pm \frac{9}{5}$. In this way the sum of the variances corresponding to the three spin components $J_\alpha = \sigma_\alpha/2$ becomes

$$\langle \Delta J^2 \rangle = \frac{1}{4} \left(3\frac{81}{25} - \cdot \right) = \frac{109}{50} . \tag{5}$$

The uncertainty for a measurement performed by projecting onto spin coherent states [6] reads [7]

$$\langle \Delta J^2 \rangle \geq 2j - 1 , \tag{6}$$

where for $j = 1/2$ and pure states the bound is achieved. and is equal to 2. So, the joint measurement via universally covariant cloning does not achieve the minimum added noise. Notice also that the minimum added noise would be achieved by a discrete POVM of the form $\Pi(\vec{m}) = \frac{1}{8}[\mathbf{1} + \vec{m} \cdot \vec{\sigma}]$, with $m_\alpha = \pm 1$.

Cloning for harmonic oscillator and joint measurement of conjugated variables

We consider now the $1 \to 2$ cloning transformation for a bosonic system

$$T(\varrho) = \frac{1}{2} P_{c,a}(\sigma)(\varrho \otimes 1_a) P_{c,a}(\sigma) , \tag{7}$$

where ϱ denotes the initial state of the system to be cloned (in the bosonic mode c), mode a supports the second copy, and the projector $P_{c,a}(\sigma)$ is given by [7]

$$P_{c,a}(\sigma) = S_c(\ln \sigma) \otimes S_a(\ln \sigma) V(|0\rangle_{c\,c}\langle 0| \otimes 1_a) V^\dagger S_c^\dagger(\ln \sigma) \otimes S_a^\dagger(\ln \sigma) \tag{8}$$

with $|0\rangle$ denoting the vacuum state, $S_d(r) = \exp[r(d^{\dagger 2} - d^2)/2]$, and $V = \exp[\frac{\pi}{4}(c^\dagger a - c a^\dagger)]$. The cloning transformation in Eq. (7) can be realized by a unitary interaction of modes c and a with an ancillary system, as shown in Ref. [8]. An experimental realization of this continuous variable cloning has been proposed in Ref. [9], where the clones correspond to single-mode radiation fields and the cloning machine is a network of three parametric amplifiers under suitable gain conditions.

Upon measuring the quadrature operators $X_c = (c + c^\dagger)/2$ and $Y_a = (a - a^\dagger)/2i$ over the clones—namely projecting the output copies on the eigenstates $|x\rangle_c$ and $|y\rangle_a$—one implements the following POVM

$$
\begin{aligned}
F_\sigma(x,y) &= \frac{1}{2} \mathrm{Tr}_a[P_{c,a}(\sigma)|x\rangle_{c\,c}\langle x| \otimes |y\rangle_{a\,a}\langle y| P_{c,a}(\sigma)] \\
&= \frac{1}{\pi} D_c(x + iy) S_c(\ln \sigma)|0\rangle_{c\,c}\langle 0 S_c^\dagger(\ln \sigma) D_c^\dagger(x + iy) ,
\end{aligned}
$$

where $D_c(\alpha) = \exp(\alpha c^\dagger - \alpha^* c)$ denotes the displacement operator for mode c. Such kind of POVM is formally a squeezed-coherent state, and it provides the optimal joint measurement of the two noncommuting quadrature operators $X_{\pm\phi} = (c^\dagger e^{\pm i\phi} + c e^{\mp i\phi})$, with $\phi = \mathrm{arctg}(\sigma^2)$. This is shown by the relations

$$\int dx \int dy \, (x \cos \phi \pm y \sin \phi) F_\sigma(x,y) = X_{\pm\phi} , \tag{9}$$

$$\int dx \int dy \, (x \cos \phi \pm y \sin \phi)^2 F_\sigma(x,y)$$

$$= X_{\pm\phi}^2 + \frac{1}{4}|\sin(2\phi)| = X_{\pm\phi}^2 + \frac{1}{2}|[X_\phi, X_{-\phi}]| , \tag{10}$$

namely the outcomes $(x \cos \phi \pm y \sin \phi)$ have the same average values as the expectations of the observables $X_{\pm\phi}$ respectively, with minimum added noise.

The cloning map in Eq. (7) is *not* universally covariant, but is covariant only under the group of unitary displacement operators, namely

$$T\left(D_c(\alpha) \varrho D_c^\dagger(\alpha)\right) = D_c(\alpha) \otimes D_a(\alpha) T(\varrho) D_c^\dagger(\alpha) \otimes D_a^\dagger(\alpha) . \tag{11}$$

Conclusions

Measures of quality other than fidelity should be used for optimality of quantum cloning, depending on the final use of the output copies. We have shown that universally covariant cloning—which maximizes the fidelity— is not optimal for engineering novel schemes of joint measurement of noncommuting observables. If one wants to use quantum cloning to achieve joint measurements, cloning must be optimized for a reduced covariance group, as shown here in the case of harmonic oscillator.

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