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Multiboson Holstein-Primakoff squeezed states for SU(2) and SU(1,1)

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We define a new set of squeezed states using group-theoretical methods. The definition is based on the Holstein-Primakoff realization of both SU(2) and SU(1,1). Generalizations of these realizations are presented, connected with the Brandt-Greenberg generalized Bose operators. The new states exhibit interesting squeezing properties, depending in a characteristic way on the dimension of the irreducible unitary representation adopted. We also discuss the asymptotic behavior and present a set of relevant numerical results. Unexpected and interesting scaling behavior appears.

I. INTRODUCTION

Glauber's coherent states[1] correspond, in configuration space, to minimum-uncertainty Gaussian wave packets whose width is everywhere equal to that of the vacuum state, produced when a harmonic oscillator interacts with a classical field of force.

In view of the problem of detecting effects close to the sensitivity limits imposed by quantum mechanics (such as gravitational radiation[2] or multiphoton eigenmodes of the electromagnetic field in an optical cavity[3]), a different set of states has been proposed,[4] which in configuration space correspond to Gaussian wave packets with widths distorted from that of the vacuum. The latter are referred to as squeezed states. Squeezed states should indeed be thought of as states obtained by displacing a squeezed vacuum by the same displacement operator which generates Glauber's coherent states. However, since all the relevant information and features are contained in such a squeezed vacuum, it is the latter which is more often referred to as a squeezed state.

In a recent paper by Fisher, Nieto, and Sandberg[5] the concept of squeezed coherent states of a harmonic oscillator was thoroughly reviewed in view of a possible higher-order generalization of the squeezing operator which might lead to new distribution functions and allow a different nonlinear detection scheme resorting to multiphoton states.

The result was negative in the sense examined by Fisher *et al.* in that it yielded operators leading to a nonanalytic ground state. However, D'Ariano, Rasetti, and Vadacchino[6] produced a different generalization, resorting to generalized k-boson operators, which corresponds to non-Gaussian[7] multiphoton squeezed states.

It is of interest to mention here that the customary squeezed states can be viewed as coherent states for SU(1,1) in the framework of the concept of generalized coherent states for an arbitrary Lie group.

In this paper we rely on this group-theoretical approach to define a new set of highly nontrivial generalized squeezed states which in suitable limits reproduce both the usual squeezed states and those of Ref. 6. In addition we study the squeezing properties of these states as well as the relation between the different realizations and the different limits.

From the group-theoretical point of view these generalized squeezed states are connected with both SU(2) and SU(1,1) in their Holstein-Primakoff realizations.

In the SU(2) case the number of photons is finite, and there ensues an interesting lower bound on the amount of squeezing one can possibly achieve.

The notation used throughout the paper is standard. In Sec. II the multiboson Holstein-Primakoff coherent states for SU(2) are defined, and their relation to the generalized Bose operators is analyzed. In Sec. III the squeezing properties of such states are thoroughly discussed in general whereas Sec. IV is devoted to their asymptotic behavior. In Sec. V both the definitions and the analysis are carried through to the SU(1,1) case. Section VI contains a collection of numerical results, meant both to check the analytical results of the previous sections and to clarify some global behavior. Some unexpected scaling properties of the optimal squeezing with respect to the unitary irreducible representation label appear. A summary and some further discussion are presented in Sec. VII.

II. MULTIBOSON HOLSTEIN-PRIMAKOFF RELATIONS FOR SU(2)

The Holstein-Primakoff[8] relations form a realization of the SU(2) algebra in terms of a single Bose operator (notice that wherever the functions of the operator $\hat{n}$ occur, it is implied that these functions are only evaluated in eigenstates of $\hat{n}$ and therefore considered equal to the functions of the eigenvalue; the commutation relations are obtained in the same spirit):

$$\begin{aligned} J_- &= (2\sigma-\hat{n})^{1/2}a \ , \\ J_+ &= J_-^\dagger = a^\dagger(2\sigma-\hat{n})^{1/2} \\ &= (2\sigma+1-\hat{n})^{1/2}a^\dagger \ , \qquad (2.1) \\ J_3 &= \tfrac{1}{2}[J_+,J_-] = \hat{n}-\sigma \ . \end{aligned}$$

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Here $[a,a^\dagger]=1$ and $\hat{n}=a^\dagger a$. The states spanning the basis for the $(2\sigma+1)$-dimensional representation of SU(2) are the normalized bosonic states

$$|n\rangle=(n!)^{-1/2}(a^\dagger)^n|0\rangle,\quad n=0,1,\ldots,2\sigma$$

with J_3 eigenvalues ranging from $-\sigma$ to σ, respectively. We may generalize the relations (2.1) as follows. Writing

$$J_-^{(k)}=a^k f_{k,\sigma}(\hat{n})=f_{k,\sigma}(\hat{n}+k)a^k\,,\qquad J_+^{(k)}=f_{k,\sigma}(\hat{n})(a^\dagger)^k \tag{2.2}$$

we obtain

$$[J_+^{(k)},J_-^{(k)}]=\frac{\hat{n}!}{(\hat{n}-k)!}f_{k,\sigma}{}^2(\hat{n})-\frac{(\hat{n}+k)!}{\hat{n}!}f_{k,\sigma}{}^2(\hat{n}+k)\,. \tag{2.3}$$

Identifying $|0\rangle$ with the eigenstate $|J_3^{(k)}=-\sigma\rangle$ the form of $J_-^{(k)}$ suggests that $J_3^{(k)}=[[\hat{n}/k]]-\sigma$ so that the complete representation is spanned by the states $|0\rangle,|k\rangle,\ldots,|2\sigma k\rangle$ corresponding to eigenvalues of $J_3^{(k)}$ equal to $-\sigma,-\sigma+1,\ldots,\sigma$, respectively. $[[x]]$ denotes the largest integer $\leq x$. Comparison of this form of $J_3^{(k)}$ with the right-hand side of Eq. (2.3) provides a recurrence relation for $f_{k,\sigma}(\hat{n})$, which can be shown to result in

$$f_{k,\sigma}(\hat{n})=((\hat{n}-k)!/\hat{n}![[\hat{n}/k]]\{2\sigma+1-[[\hat{n}/k]]\})^{1/2}\,. \tag{2.4}$$

This result could have been obtained by starting from the usual Holstein-Primakoff relations, Eqs. (2.1), and substituting the generalized Bose operators[9]

$$A_{(k)}^\dagger=f_k(\hat{n})(a^\dagger)^k\,,\qquad \hat{N}_{(k)}=A_{(k)}^\dagger A_{(k)}=[[\hat{n}/k]]\,, \tag{2.5}$$

where

$$f_k(\hat{n})=\{[[\hat{n}/k]](\hat{n}-k)!/\hat{n}!\}^{1/2}\,.$$

These generalized Bose operators satisfy the usual boson commutation relations $[A_{(k)},A_{(k)}^\dagger]=1$.

Performing the indicated substitution we obtain

$$\begin{aligned} J_+^{(k)}&=(2\sigma+1-\hat{N}_{(k)}A_{(k)}^\dagger)^{1/2}\\ &=\{2\sigma+1-[[\hat{n}/k]]\}^{1/2}f_k(\hat{n})(a^\dagger)^k\,,\\ J_-^{(k)}&=(J_+^{(k)})^\dagger\,, \end{aligned} \tag{2.6}$$

and

$$J_3^{(k)}=\hat{N}_{(k)}-\sigma=[[\hat{n}/k]]-\sigma\,.$$

Comparison of Eq. (2.6) with Eq. (2.4) gives

$$f_{k,\sigma}(\hat{n})=\{2\sigma+1-[[\hat{n}/k]]\}^{1/2}f_k(\hat{n})\,. \tag{2.7}$$

III. SQUEEZING PROPERTIES OF MULTIBOSON HOLSTEIN-PRIMAKOFF COHERENT STATES FOR SU(2)

Harmonic-oscillator coherent states[1] are quantum states having the following properties: the uncertainty product $\Delta x\Delta p$ attains its minimum value, with $\Delta x=\Delta p=1/\sqrt{2}$ ($\hbar=1$); they evolve quasiclassically and they have the form $\exp(\lambda a^\dagger)|0\rangle$. The question that now arises is the following: Can we improve upon the uncertainty $(\Delta x)^2=\frac{1}{2}$, for example, while still maintaining the quasiclassical evolution? The solution (the squeezed states) suggested by the form $\exp(\lambda a^\dagger)|0\rangle$, where $a,a^\dagger$ generate the Weyl algebra, is $\exp(\mu J_+)|0\rangle$, where J_+ is a bosonic realization of a raising operator in a more general algebra. The squeezed states are then coherent states for the group of such an algebra. We recall the construction of coherent states of SU(2) according to the general definition for an arbitrary Lie group given by Perelomov[10] and Rasetti.[11]

The set of coherent states for a Lie group G is obtained using a unitary irreducible representation (UIR) of the group, choosing a fixed vector $|\omega\rangle$ in the representation space, and acting on it by the whole group. It turns out that the coherent states are labeled by means of the left cosets of the group G with respect to the subgroup leaving $|\omega\rangle$ invariant up to a phase factor. For the SU(2) group the set of coherent states within the $(2\sigma+1)$-dimensional UIR is given by the formula

$$|\xi;\sigma\rangle=\exp[P(\xi J_+-\xi^*J_-)]\,|J_3=-\sigma;\sigma\rangle\,, \tag{3.1}$$

where ξ is a complex parameter, and $|J_3=-\sigma;\sigma\rangle$ is the eigenvector of J_3 corresponding to the eigenvalue $-\sigma$. Using the Baker-Campbell-Hausdorff formula for SU(2)

$$\exp(\xi J_+-\xi^*J_-)=e^{\zeta J_+}e^{\beta J_3}e^{-\zeta^*J_-}\,, \tag{3.2}$$

where

$$\zeta=\frac{\xi}{|\xi|}\tan(|\xi|),\quad \beta=\ln(1+|\zeta|^2)\,, \tag{3.3}$$

Eq. (3.1) can be rewritten in the form

$$|\zeta;\sigma\rangle=N^{-1}e^{\zeta J_+}|J_3=-\sigma;\sigma\rangle\,. \tag{3.4}$$

N is a normalization coefficient and we chose to label the coherent states directly in terms of ζ. In the k-boson Holstein-Primakoff realization of the UIR of SU(2), the vector $|J_3=-\sigma;\sigma\rangle$ turns out to be the vacuum and the coherent states are written in the reduced form

$$|\zeta;k,\sigma\rangle=N^{-1}e^{\zeta J_+^{(k)}}|0\rangle\,. \tag{3.5}$$

Expanding the exponential and using the properties of the $J_+^{(k)}$ operators we obtain

$$|\zeta;k,\sigma\rangle=N^{-1}\sum_{l=0}^{2\sigma}\frac{\zeta^l}{l!}\left[\prod_{p=0}^{l-1}f_{k,\sigma}(\hat{n}-pk)\right](a^\dagger)^{kl}|0\rangle\,. \tag{3.6}$$

In Eq. (3.6) the number operator $\hat{n}$ can be replaced by its eigenvalue kl. Using the explicit form of $f_{k,\sigma}(\hat{n})$, Eq. (2.4), Eq. (3.6) becomes

$$|\zeta;k,\sigma\rangle = N^{-1}\sum_{l=0}^{2\sigma}\zeta^l\left\{\begin{bmatrix}2\sigma\\ l\end{bmatrix}\right\}^{1/2}|kl\rangle\ . \tag{3.7}$$

This is the general expression for the SU(2) coherent states, where $|kl\rangle$ are the weight vectors of the $(2\sigma+1)$-dimensional representation, namely, the eigenvectors of $J_3^{(k)}$. Since

$$\langle\zeta;k,\sigma|\zeta;k,\sigma\rangle = \frac{1}{N^2}(1+|\zeta|^2)^{2\sigma} \tag{3.8}$$

it follows that

$$N=(1+|\zeta|^2)^{\sigma}\ . \tag{3.9}$$

Furthermore,

$$\langle\zeta;k,\sigma|a^\dagger a\,|\zeta;k,\sigma\rangle = \frac{2\sigma k\,|\zeta|^2}{1+|\zeta|^2} \tag{3.10}$$

and

$$\langle\zeta;k,\sigma|(a^\dagger)^{km}|\zeta;k,\sigma\rangle = \frac{\zeta^{*m}}{(1+|\zeta|^2)^{2\sigma}}\sum_{l=0}^{2\sigma}|\zeta|^{2l}\left\{\begin{bmatrix}2\sigma\\ l\end{bmatrix}\begin{bmatrix}2\sigma\\ l+m\end{bmatrix}\times\frac{[k(l+m)]!}{(kl)!}\right\}^{1/2},$$

$$\langle\zeta;k,\sigma|(a^\dagger)^{km+r}|\zeta;k,\sigma\rangle = 0\ , \quad r=1,2,\ldots,m-1,\ m=1,2,\ldots\ . \tag{3.11}$$

For $k=1$ we obtain

$$\langle a^\dagger\rangle = \frac{\zeta^*}{(1+|\zeta|^2)^{2\sigma}}\sum_{l=0}^{2\sigma}|\zeta|^{2l}\begin{bmatrix}2\sigma\\ l\end{bmatrix}\sqrt{2\sigma-l}\ , \tag{3.12}$$

$$\langle(a^\dagger)^2\rangle = \frac{\zeta^{*2}}{(1+|\zeta|^2)^{2\sigma}}\times\sum_{l=0}^{2\sigma}|\zeta|^{2l}\begin{bmatrix}2\sigma\\ l\end{bmatrix}\sqrt{(2\sigma-l)(2\sigma-l-1)}\ , \tag{3.13}$$

$$\langle a^\dagger a\rangle = \frac{2\sigma|\zeta|^2}{1+|\zeta|^2}\ , \tag{3.14}$$

from which $(\Delta x)^2=\langle x^2\rangle-\langle x\rangle^2$ and $(\Delta p)^2=\langle p^2\rangle-\langle p\rangle^2$ follow via the relations $x=(1/\sqrt{2})(a^\dagger+a)$ and $p=(i/\sqrt{2})(a^\dagger-a)$:

$$(\Delta x)^2=\tfrac{1}{2}+\langle a^\dagger a\rangle-\langle a^\dagger\rangle\langle a\rangle+\mathrm{Re}[\langle(a^\dagger)^2\rangle-\langle a^\dagger\rangle^2]\ ,$$
$$(\Delta p)^2=\tfrac{1}{2}+\langle a^\dagger a\rangle-\langle a^\dagger\rangle\langle a\rangle-\mathrm{Re}[\langle(a^\dagger)^2\rangle-\langle a^\dagger\rangle^2]\ . \tag{3.15}$$

For $k=2$,

$$\langle a^\dagger\rangle=\langle a\rangle=0\ ,$$
$$\langle a^\dagger a\rangle=\frac{4\sigma|\zeta|^2}{1+|\zeta|^2}\ , \tag{3.16}$$
$$\langle(a^\dagger)^2\rangle=\frac{\sqrt{2}\zeta^*}{(1+|\zeta|^2)^{2\sigma}}\times\sum_{l=0}^{2\sigma}|\zeta|^{2l}\begin{bmatrix}2\sigma\\ l\end{bmatrix}\sqrt{(2\sigma-l)(2l+1)}\ .$$

For $k\geq 3$,

$$\langle a^\dagger\rangle=\langle a\rangle=0,\quad \langle(a^\dagger)^2\rangle=\langle a^2\rangle=0\ ,$$
$$\langle a^\dagger a\rangle=\frac{2\sigma k\,|\zeta|^2}{1+|\zeta|^2}\ .$$

IV. ASYMPTOTIC BEHAVIOR

Since, by Eq. (3.10),

$$\langle\zeta;k,\sigma|[[\hat{n}/k]]|\zeta;k,\sigma\rangle=\frac{2\sigma|\zeta|^2}{1+|\zeta|^2} \tag{4.1}$$

it is reasonable to investigate the limiting forms of the results presented in the previous section in the limit in which $\sigma\to\infty$ such that $\rho=\sqrt{2\sigma}|\zeta|$ remains finite. Clearly,

$$\langle[[\hat{n}/k]]\rangle=\frac{\rho^2}{1+\frac{\rho^2}{2\sigma}}\simeq\rho^2\left[1-\frac{\rho^2}{2\sigma}\right] \tag{4.2}$$

and

$$N=\left[1+\frac{\rho^2}{2\sigma}\right]^{\sigma}\simeq e^{\rho^2/2}\left[1-\frac{\rho^4}{8\sigma}+\cdots\right]\ . \tag{4.3}$$

In the same limit we have $(2\sigma+1-[[\hat{n}/k]])^{1/2}\simeq\sqrt{2\sigma}$, so that

$$J_+^{(k)}\to\sqrt{2\sigma}A_{(k)}^\dagger,\quad J_-^{(k)}\to\sqrt{2\sigma}A_{(k)} \tag{4.4}$$

and the k-boson Holstein-Primakoff coherent states reduce to the generalized k-boson coherent states. This result is well known in the case $k=1$ corresponding to the standard Holstein-Primakoff coherent states reducing to the Glauber coherent states.[12] Retaining terms up to order $1/\sigma$ in Eq. (3.11) we obtain, selecting for simplicity ζ to be real and positive,

$$\langle\zeta;k,\sigma|(a^\dagger)^{km}|\zeta;k,\sigma\rangle\simeq e^{-\rho^2}\rho^m\left[1+\frac{\rho^4}{4\sigma}\right]\sum_{l=0}^{\infty}\rho^{2l}\left[\frac{[k(l+m)]!}{l!(l+m)!(kl)!}\right]^{1/2}\left[1-\frac{1}{8\sigma}[l(l-1)+(l+m)(l+m-1)]\right]\ . \tag{4.5}$$

Some interesting special cases are $k=1$,

$$\langle a^\dagger\rangle \simeq \rho - \frac{\rho^3}{4\sigma} , \tag{4.6}$$

$$\langle (a^\dagger)^2\rangle \simeq \rho^2 - \frac{1}{4\sigma}(\rho^2+2\rho^4) ; \tag{4.7}$$

and $k=2$,

$$\langle (a^\dagger)^2\rangle \simeq \sqrt{2}F_1(\rho^2) + \frac{1}{2\sqrt{2}\sigma}[\rho^4 F_1(\rho^2) - F_3(\rho^2) - F_5(\rho^2)] , \tag{4.8}$$

where

$$F_n(x) = x^{n/2}e^{-x}\sum_{l=0}^{\infty}\frac{x^l}{l!}\sqrt{2l+n} . \tag{4.9}$$

Note that $F_1(\rho^2)=\rho F(\rho^2)$, where $F(\rho^2)$ is defined in Ref. 6. Thus, for $k=1$ we obtain

$$(\Delta x)^2 \simeq \frac{1}{2} - \frac{\rho^2}{4\sigma} , \quad (\Delta p)^2 \simeq \frac{1}{2} + \frac{\rho^2}{4\sigma} . \tag{4.10}$$

These results imply that to first order in $1/\sigma$ the uncertainty product remains minimal. Thus, for large but finite σ we obtain near-minimum uncertainty and nonzero squeezing. For $k=2$

$$(\Delta x)^2 \simeq \tfrac{1}{2} + 2\rho^2 + \sqrt{2}F_1(\rho^2) - \frac{1}{\sigma}\left[\frac{1}{2\sqrt{2}}[F_3(\rho^2)+F_5(\rho^2)-\rho^4F_1(\rho^2)]+\rho^4\right] , \tag{4.11}$$

$$(\Delta p)^2 \simeq \tfrac{1}{2} + 2\rho^2 - \sqrt{2}F_1(\rho^2) + \frac{1}{\sigma}\left[\frac{1}{2\sqrt{2}}[F_3(\rho^2)+F_5(\rho^2)-\rho^4F_1(\rho^2)]-\rho^4\right] . \tag{4.12}$$

Equation (4.12) results in squeezing of $(\Delta p)^2$. Had we selected ζ negative we would have obtained squeezing of $(\Delta x)^2$. For each finite σ the value of $\rho_{\min}$ corresponding to the optimally squeezed $(\Delta p)^2{}_{\min}$ is given by differentiation of (4.12). Writing

$$\rho_{\min} \simeq \rho_0 + \frac{\gamma}{\sigma} + O(1/\sigma^2) \tag{4.13}$$

we find that ρ_0 is the solution of the zeroth-order equation:

$$4\rho_0 - \sqrt{2}\left.\frac{\partial F_1}{\partial \rho}\right|_{\rho=\rho_0} = 0 . \tag{4.14}$$

γ is given by

$$\gamma = \left.\frac{\partial B}{\partial \rho}\right|_{\rho=\rho_0} \Bigg/ \left[\sqrt{2}\left.\frac{\partial^2 F_1}{\partial \rho^2}\right|_{\rho=\rho_0} - 4\right] , \tag{4.15}$$

where $B(\rho^2)$ is the coefficient of $1/\sigma$ in Eq. (4.12). Furthermore,

$$(\Delta p)^2{}_{\min} \simeq \tfrac{1}{2} + 2\rho_0 - \sqrt{2}F_1(\rho_0^2) + \frac{1}{\sigma}B(\rho_0^2) . \tag{4.16}$$

From the definition of $F_n(\rho^2)$, Eq. (4.9), it follows that

$$\frac{\partial F_n(\rho^2)}{\partial \rho} = \left[\frac{n}{\rho} - 2\rho\right]F_n(\rho^2) + \frac{2}{\rho}F_{n+2}(\rho^2) \tag{4.17}$$

so that

$$\frac{\partial B}{\partial \rho} = \frac{1}{2\sqrt{2}\rho}[\rho^4(2\rho^2-5)F_1 - (2\rho^4+2\rho^2-3)F_3 - (2\rho^2-7)F_5 + 2F_7] - 4\rho^3 . \tag{4.18}$$

Solving Eq. (4.14) numerically we obtain

$$\rho_0 \simeq 0.646\,75 . \tag{4.19}$$

Substituting this value in Eq. (4.15) we get

$$\gamma \simeq -0.2294 . \tag{4.20}$$

Finally, from Eq. (4.16) we find

$$(\Delta p)^2{}_{\min} \simeq 0.158\,72 + 0.054\,73\frac{1}{\sigma} . \tag{4.21}$$

The values of ρ_0 and of $(\Delta p)^2{}_{\min}$ ($\sigma=\infty$) are just the values corresponding to maximal squeezing for the generalized boson $k=2$ coherent states of Ref. 6.

V. MULTIBOSON HOLSTEIN-PRIMAKOFF COHERENT STATES OF SU(1,1)

The commutation relations of SU(1,1)

$$[J_3, J_\pm] = \pm J_\pm , \quad [J_+, J_-] = -2J_3 \tag{5.1}$$

can be realized by expressing the generators $J_\pm$ and J_3 in terms of multiboson operators.

Thus, in the UIR corresponding to the eigenvalue $(\sigma^2-\sigma)$ of the Casimir operator $J_3{}^2 - \frac{1}{2}(J_+J_- + J_-J_+)$ we have

$$\begin{aligned} J_+^{(k)} &= (2\sigma - 1 + A_{(k)}^\dagger A_{(k)})^{1/2}A_{(k)}^\dagger , \\ J_-^{(k)} &= [J_+^{(k)}]^\dagger = A_{(k)}(2\sigma - 1 + A_{(k)}^\dagger A_{(k)})^{1/2} , \\ J_3^{(k)} &= [J_3^{(k)}]^\dagger = A_{(k)}^\dagger A_{(k)} + \sigma . \end{aligned} \tag{5.2}$$

Note that the representations labeled by σ are now infinite dimensional. The special case $k=1$ was considered by Gerry.[13] The SU(1,1) coherent states are defined as

$$|z;\sigma\rangle = e^{zJ_+ - z^*J_-}|J_3=\sigma;\sigma\rangle . \tag{5.3}$$

Using the Baker-Campbell-Hausdorff formula we obtain

$$|\alpha;\sigma\rangle = \frac{1}{\mathcal{N}}e^{\alpha J_+}|J_3=\sigma;\sigma\rangle , \tag{5.4}$$

where $\alpha=(z/|z|)\tanh(|z|)$ and $|\alpha,\sigma\rangle$ is normalized. In complete analogy with the SU(2) analysis we obtain

$$|\alpha;k,\sigma\rangle = \frac{1}{\mathcal{N}}\sum_{m=0}^{\infty}\alpha^m\left\{\begin{bmatrix}2\sigma+m-1\\ m\end{bmatrix}\right\}^{1/2}|km\rangle , \tag{5.5}$$

$$\mathcal{N}=(1-|\alpha|^2)^{-\sigma}\,, \tag{5.6}$$

and

$$\langle \alpha;k,\sigma\,|\,a^\dagger a\,|\,\alpha;k,\sigma\rangle=\frac{2k\sigma|\alpha|^2}{1-|\alpha|^2}\,. \tag{5.7}$$

Furthermore, for $k=1$

$$\langle a^\dagger\rangle=\frac{\alpha^*}{\mathcal{N}^2}\sum_{m=0}^{\infty}|\alpha|^{2m}\begin{bmatrix}2\sigma+m-1\\ m\end{bmatrix}\sqrt{2\sigma+m}\ , \tag{5.8}$$

$$\langle (a^\dagger)^2\rangle=\frac{\alpha^{*2}}{\mathcal{N}^2}\sum_{m=0}^{\infty}|\alpha|^{2m}\begin{bmatrix}2\sigma+m-1\\ m\end{bmatrix}\times\sqrt{(2\sigma+m)(2\sigma+m+1)}\ , \tag{5.9}$$

and for $k=2$

$$\langle (a^\dagger)^2\rangle=\frac{\sqrt{2}\alpha^*}{\mathcal{N}^2}\sum_{m=0}^{\infty}|\alpha|^{2m}\begin{bmatrix}2\sigma+m-1\\ m\end{bmatrix}\times\sqrt{(2\sigma+m)(2m+1)}\ . \tag{5.10}$$

The position and momentum uncertainties are obtained in terms of the above matrix elements. The definition of $J_+^{(k)}$ for SU(1,1), Eq. (5.2), suggests two interesting limits: (a) $\sigma\to\infty$ with $\langle A_{(k)}^\dagger A_{(k)}\rangle$ finite, and (b) σ finite with $\langle A_{(k)}^\dagger A_{(k)}\rangle\to\infty$.

The limit of the first kind, if taken in such a way that $\rho=|\alpha|\sqrt{2\sigma}$ remains finite, results in

$$|\alpha;k,\sigma\rangle\to\frac{1}{\mathcal{N}}e^{\rho A_{(k)}^\dagger}|0\rangle\ , \tag{5.11}$$

which is again the generalized boson coherent state.[6] Note that in this limit $\langle a^\dagger a\rangle=k\rho^2$, which is finite, as required.

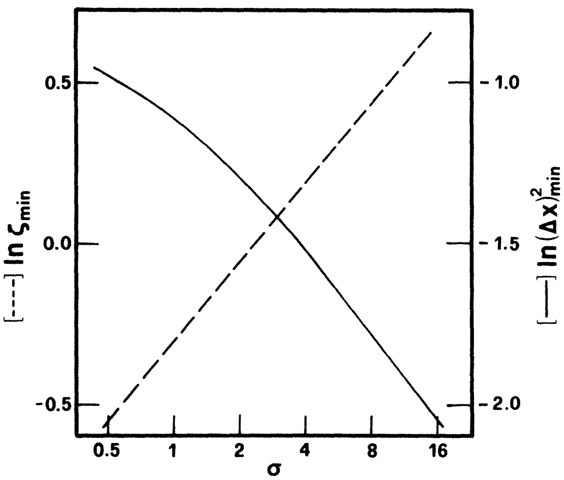

FIG. 2. Behavior of the squeezing parameter ζ and of the position uncertainty for the most-squeezed one-boson SU(2) Holstein-Primakoff states.

The limit of the second kind requires that $\alpha\to 1$. For $k=2$ we obtain, in this limit,

$$J_+^{(2)}\to\tfrac{1}{2}(a^\dagger)^2\ , \tag{5.12}$$

so that $|\alpha\rangle$ becomes a particular harmonic-oscillator squeezed state.[5] The $\sigma\to\infty$ asymptotic expressions are obtained in complete analogy with the SU(2) case. Upon computing them one notices that the results for $(\Delta x)^2$ and $(\Delta p)^2$, both for $k=1$ and $k=2$, are obtained from the expressions given in Eqs. (4.10)–(4.13) for SU(2) by reversing the sign of the coefficients of $1/\sigma$.

VI. NUMERICAL RESULTS

To illustrate and amplify the previous discussion we present the results of some numerical computations.

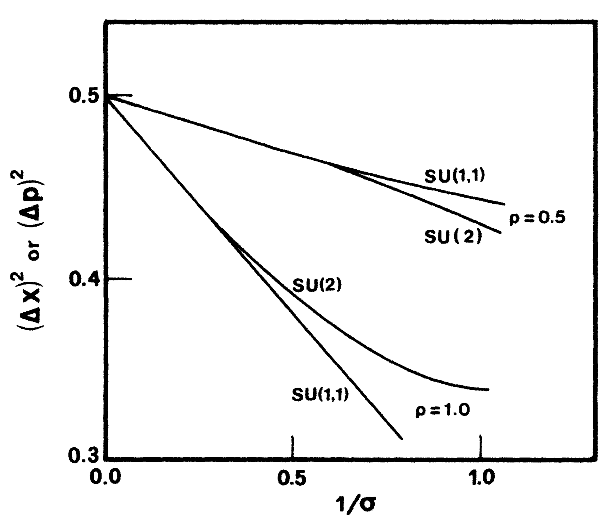

FIG. 1. Position uncertainties for one-boson SU(2) Holstein-Primakoff states with a constant ρ, and momentum uncertainties for the corresponding SU(1,1) states.

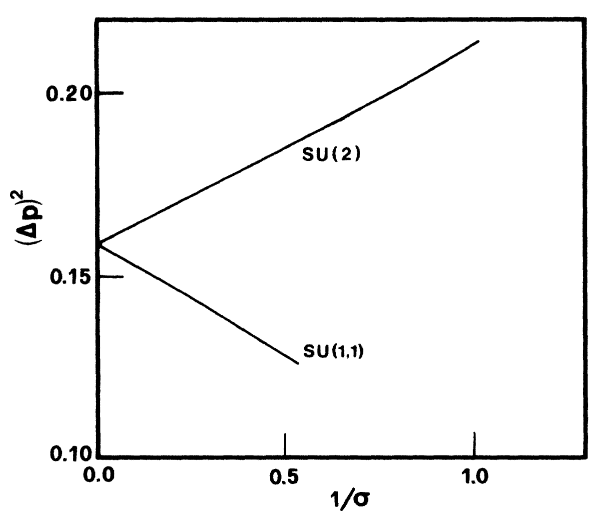

FIG. 3. The most-squeezed momentum uncertainties for the two-boson Holstein-Primakoff states [SU(2) and SU(1,1)].

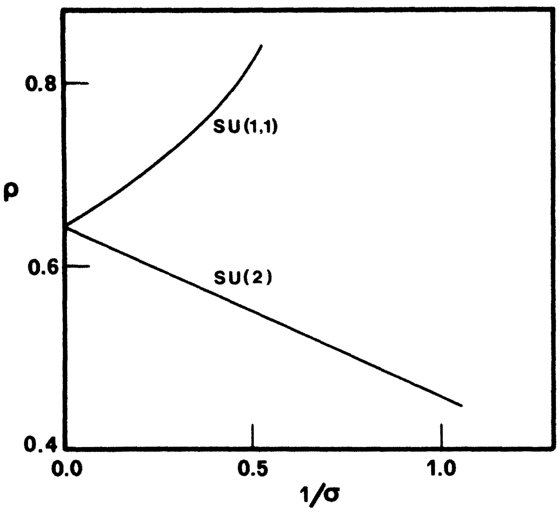

FIG. 4. The squeezing parameter for the most-squeezed two-boson Holstein-Primakoff states [SU(2) and SU(1,1)].

A. Single-boson case ($k=1$)

Computing $(\Delta x)^2$ or $(\Delta p)^2$ vs σ, for a constant value of ρ, we obtain the behavior presented in Fig. 1, exhibiting an approach to the limiting value corresponding to an unsqueezed harmonic-oscillator coherent state, as $\sigma\rightarrow\infty$.

The $(\Delta x)^2$ vs $1/\sigma$ curves, for constant ρ, indicate a close-to-linear relation with the intercept and asymptotic slope in agreement with Eq. (4.10). Figure 2 shows that the optimal squeezing of $(\Delta x)^2$ increases with σ, with $(\Delta x)^2_{\min}\rightarrow 0$ for large σ. Also shown in Fig. 2 is the optimal value of the squeezing parameter $\zeta_{\min}$. Figure 2 indicates that $(\Delta x)^2_{\min}\sim\sigma^{-\delta}$ and $\zeta_{\min}\sim\sigma^{\delta}$ where δ is approximately equal to $\frac{1}{3}$ in the SU(2) case.

B. Two-boson case ($k=2$)

Calculating the maximum squeezing of $(\Delta p)^2$ as a function of ρ, for different values of σ, we obtain the results presented in Fig. 3. The corresponding values of ρ are presented in Fig. 4. Note that both $(\Delta p)^2$ and ρ are very close to being linear in $1/\sigma$ over the range presented. The intercepts of both curves with the $1/\sigma=0$ axis, and their slopes, agree with the analytically derived values, Eqs. (4.19)–(4.21). Note, further, that for $\sigma\rightarrow\infty$ both the SU(2) and the SU(1,1) curves approach the generalized boson values.[6]

VII. CONCLUSIONS

The new set of generalized squeezed states defined and investigated in the present paper are actually the group-theoretical coherent states of the SU(2) and SU(1,1) Lie algebras. The Holstein-Primakoff realizations of these algebras in terms of the harmonic-oscillator ladder operators and, in particular, the realization introduced in the present paper in terms of multiboson operators, enabled the investigation of the squeezing properties of the group-theoretical coherent states with respect to the harmonic-oscillator dynamical variables.

In both the SU(2) and SU(1,1) cases, the single-boson and two-boson Holstein-Primakoff states exhibit squeezing of either position or momentum, depending on the phase of the squeezing parameter. However, for a Hamiltonian of the form $H=\omega a^{\dagger}a+\text{const}$, we have, for a real squeezing parameter,

$$[\Delta x(t)]^2=(\Delta x)^2\cos^2(\omega t)+(\Delta p)^2\sin^2(\omega t)\ ,$$

with a similar expression for $[\Delta p(t)]^2$, and so the squeezing oscillates between $(\Delta x)^2$ and $(\Delta p)^2$ with frequency 2ω.

For finite values of the label of the irreducible representation σ the maximum amount of squeezing is finite; i.e., neither $(\Delta x)^2$ nor $(\Delta p)^2$ can shrink indefinitely.

The optimal squeezing of the single-boson SU(2) exhibits an interesting asymptotic behavior, i.e., $(\Delta x)^2\approx\sigma^{-1/3}$. Thus, arbitrarily high squeezing can be achieved by increasing σ to sufficiently large values. Note that σ can be interpreted as the maximum number of photons available, N_{ph}, which is certainly finite in any conceivable experimental setup. Thus, in this case the presently proposed states have one physically derivable property the familiar Glauber coherent states do not have: i.e., they presuppose a finite rather than infinite number of photons. From the asymptotic relation mentioned above we obtain an increasing lower bound on the amount of squeezing achievable, i.e., $(\Delta x)^2\geq cN_{\text{ph}}^{-1/3}$, where c is some universal constant, which according to our numerical results is of order unity. While this result is very preliminary, it suggests an interesting approach to the general analysis of the ultimate limits of attainable squeezing. For the two-boson realizations of both SU(2) and SU(1,1) the optimally squeezed states approach the optimally squeezed generalized two-boson squeezed states of Ref. 6 in the limit $\sigma\rightarrow\infty$. Actually, by approaching that limit with a sequence of squeezed states such that the parameter $\rho=\zeta\sqrt{2\sigma}$ is an (arbitrary) constant, we obtain the Glauber coherent states in the single-boson case and the generalized boson coherent states in general. The asymptotic approach to this limit was studied analytically, to order $1/\sigma$. Note that the asymptotic expressions obtained for SU(1,1) are related to those obtained for SU(2) by a reversal of sign of the coefficient of $1/\sigma$. These asymptotic expressions were confirmed numerically for both SU(2) and SU(1,1) in the cases of the single-photon as well as of the two-photon Holstein-Primakoff states.

The limit considered above, in which ρ is kept constant, involves a finite expectation value of the number operator. On the other hand, the form of the two-boson Holstein-Primakoff SU(1,1) operator suggests that a different kind of limit could be defined, involving $\hat{n}$ becoming arbitrarily large. In this limit the two-boson operator J_+ attains the form $\frac{1}{2}(a^{\dagger})^2$, suggesting that the corresponding SU(1,1) coherent states become the usual harmonic-oscillator squeezed states. Note that for $k\neq 2$ this limit does not exist, which may be related to the results presented by Fisher, Nieto, and Sandberg[5] concerning the impossibility of a naive generalization of squeezing.

The results of the present paper suggest several further avenues of investigation. Among these, the most straightforward seems to be a definition of k-boson Holstein-Primakoff coherent states in which the vacuum state $|0\rangle$ is replaced by a linear combination of the form $\sum_{l=0}^{k-1} c_l |l\rangle$. Clearly, such a definition will involve the whole Fock space rather than the subspace $\{|kl\rangle; l=0,1,\ldots\}$ involved in the results presented above. This will result in much higher flexibility, including the possibility to obtain squeezing for any k.

This, as well as an analysis of the squeezing properties of higher moments of the dynamical variables, will be considered in the future.

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