From: SQUEEZED AND NONCLASSICAL LIGHT
Edited by P. Tombesi and E. R. Pike
(Plenum Publishing Corporation, 1989)

MULTIPHOTON AND FRACTIONAL-PHOTON SQUEEZED STATES

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1. Introduction

There are several motivations, both physical and mathematical to pursue the construction of multi-photon squeezed states. Besides describing many photon processes which are more and more interesting from the point of view of quantum optics, they lead to non-Gaussian vawepackets which are of great relevance for other branches of physics as well (for instance the theory of phase transitions or the description of certain collective effects in nuclei). Moreover, since in several cases higher order moments can be independently squeezed, such states depend very often on a larger number of parameters and a finer tuning of the related probability distributions can be achieved.

The origin of the set of papers ([1], [2], [3], [4], [5], [6], [7], [8], [9], [10]) which motivated the present work, and which are in part briefly reviewed in it, was the puzzling paper by Fisher, Nieto and Sandberg [11]. In it the authors, trying to generalize the customary 2-photon squeezed states of Stoler and Yuen [12][13] to multiphoton states by the simplest possible ansatz, run into unexpected difficulties connected with the non-analyticity of the vacuum. Even though such difficulties could be partly overcome from the computational point of view (Ref. [14]), the problem is a very deep one.

A *non-naïve* way out of it was found on the basis and in terms of a number of observations : *i*) it is straightforward to check that the conventional squeezed states are generalized coherent states (in the sense of [15]), corresponding to the algebra $su(1,1)$. (In this framework they naturally fit into the general definition given by Glauber [16]). *ii*) The generalization proposed leads to an infinite-dimensional algebra whose coherent states are unknown – indeed they are most probably undefinable in the usual sense – and which is anyway endowed with a structure not rigid enough for such a fine effect as squeezing. *iii*) One should have therefore to preserve the two ingredients which appear to control the whole phenomenon : on the one hand the Weyl-Heisenberg (W.H.) group skeleton, responsible for the bosonic character of the many photon states, on the other the structure of a group, compact or non-compact but of finite rank, to generate squeezing. *iv*) The price one should be ready to pay is the recourse to non-linear realizations of the algebra, which necessarily introduce into

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play infinite power series of bosonic operators (rigid enough though not to break the delicate balance leading to squeezing).

The two main tools to realize the above program are the Brandt-Greenberg [17] multiphoton creators and annihilators, and the Holstein-Primakoff (H.P.) [18] realization of the $su(2)$ and $su(1,1)$ algebras.

The new set of multiphoton squeezed states thus constructed has several interesting features. They promise to be the best candidates as the quantum states in which the light entering the input ports of devices such as the conventional Mach-Zehnder and Fabry-Perot interferometers, or the active lossless interferometers of Yurke, McCall and Klauder should be prepared. It was in fact shown in [19] that such devices can be quite naturally characterized by an action on a Lie group space, respectively of $SU(2)$ and $SU(1,1)$, and we expect that the states described here should be able to achieve better phase sensitivity than the highest weight vector states the authors in [19] propose.

Moreover they constitute a family of quantum states whereby, with great flexibility one can achieve an arbitrary reduction of the photon number noise, a variety of different number of photon distribution laws, a finer tuning in the control of higher moments squeezing.

Finally they lead to the notion, discussed in this paper with some detail, of fractional photon states. These are mixed states, to be realized in terms of suitable density matrices , which describe the same physical output that one should have were one able to generate photon states corresponding to a fractional eigenvalue of the number operator ([8]). Indeed they describe synthetically the complex canonical transformation and projection operation (in Hilbert space of states) one should perform when describing the observable properties of a k-photon dynamical variable in the framework of a k'-photon state, when k and k' are not multiple of one another. We show here (see Ref. [10]) that in this description squeezing corresponds to fractioning, a suggestive physical image of this elusive phenomenon.

The paper is organized as follows. In Sect. 2 we briefly review the algebraic background necessary for the theory, namely the automorphism of the W.H. algebra realized by the multiphoton operators, and both the H.P. realization and the multiphoton H.P. realization of $SU(2)$ and $SU(1,1)$. In Sect. 3 we describe the whole set of new states which can thus be obtained, and discuss the related probability distribution functions, higher order moments and squeezing properties. There appear a set of very interesting scaling properties, which exhibit unexpected universality features. In Sect. 4 we introduce the notion of fractional photon states, and discuss the possibility of their physical realization. A few conclusive comments are given in Sect. 5.

2. Algebraic background

2.1 Multiphoton operators

The new type of multiphoton squeezing operators is constructed resorting to the generalized Bose operators of Brandt and Greenberg [17] $b_{(k)}$ and $b^{\dagger}_{(k)}$. The latter satisfy the commutation relations

$$[b_{(k)}, b^{\dagger}_{(k)}] = 1 \quad , \tag{2.1}$$

$$[N, b_{(k)}] = -k b_{(k)} \quad , \tag{2.2}$$

where $N = a^{\dagger}a$ is the usual number operator.

Equations (2.1) and (2.2) lead to interpreting $b_{(k)}$ and $b^{\dagger}_{(k)}$ as annihilation and creation operators of k photons simultaneously, even though it should be noted that $b_{(1)} = a$, but $b_{(k)} \neq a^k$ for $k \geq 2$.

From (2.1) and (2.2) one can derive the normal-ordered representation

$$b_{(k)} = \sum_{j=0}^{\infty} \alpha_j^{(k)} (a^{\dagger})^j a^{j+k} \quad , \tag{2.3}$$

where

$$\alpha_j^{(k)} = \sum_{l=0}^{j} \frac{(-)^{j-l}}{(j-l)!} \left(\frac{1 + [[\frac{l}{k}]]}{l!(l+k)!} \right)^{\frac{1}{2}} e^{i\phi_l} \quad . \tag{2.4}$$

In (2.4) $[[x]]$ denotes the maximum integer $\geq x$, whereas the phases $\phi_m, m = 0, \ldots, j$ are arbitrary real numbers.

In the Fock space $b_{(k)}$ and $b_{(k)}^\dagger$ operate as follows:

$$b_{(k)}|sk + \lambda\rangle = \sqrt{s}|sk + \lambda\rangle \quad , \tag{2.5a}$$

$$b_{(k)}^\dagger|sk + \lambda\rangle = \sqrt{s+1}|(s+1)k + \lambda\rangle \quad ; \tag{2.5b}$$

where $0 \leq \lambda \leq k - 1$.

One can notice from (2.5) that the Fock space splits into k orthogonal subspaces which are invariant under the action of the k-photon operators:

$$\begin{gathered} \mathcal{F} = \bigoplus_{\lambda=0}^{k-1} \mathcal{F}_\lambda^{(k)} \quad , \quad \mathcal{F}_\lambda^{(k)} = \bigoplus_{s=0}^{\infty} span\{|sk + \lambda\rangle\} \quad , \\ b_{(k)}\mathcal{F}_\lambda^{(k)} \subset \mathcal{F}_\lambda^{(k)} \quad , \quad b_{(k)}^\dagger \mathcal{F}_\lambda^{(k)} \subset \mathcal{F}_\lambda^{(k)} \quad . \end{gathered} \tag{2.6}$$

The generic Fock state $|sk + \lambda\rangle$ is thus labeled by two quantum numbers s and λ, which are the eigenvalues of the complete set of commuting operators $b_{(k)}^\dagger b_{(k)}$ and $\hat{D}_{(k)} = a^\dagger a - k b_{(k)}^\dagger b_{(k)}$:

$$b_{(k)}^\dagger b_{(k)}|sk + \lambda\rangle = s|sk + \lambda\rangle \quad , \tag{2.7a}$$

$$\hat{D}_{(k)}|sk + \lambda\rangle = \lambda|sk + \lambda\rangle \quad . \tag{2.7b}$$

In Sec. 4 we shall equivalently consider a different set of commuting operators, namely $\hat{D}_{(k)}$ itself together with the canonical operator $\hat{Q}_{(k)}$

$$\hat{Q}_{(k)} = \frac{1}{\sqrt{2}}(b_{(k)} + b_{(k)}^\dagger) \quad . \tag{2.7c}$$

2.2 Holstein-Primakoff realizations of Lie algebras

The new set of squeezed states is defined by means of multiboson realizations of Lie algebras. In the papers [2-10] almost all simple Lie algebras and the usual solvable Weyl-Heisenberg algebra defined in (2.1) have been considered.

According to the Levi theorem, all these algebras are essentially the building blocks of every Lie algebra: this means that we can deal with a generic Lie algebra by decomposing it into its fundamental blocks. On the other hand in ref. [9] it is proved that the main squeezing properties for higher rank $SU(n)$ states reduce to those of $SU(2)$ and one can recover all the interesting features limiting the attention to the lowest rank Lie algebras. We thus consider only $W.H.$ (Weyl-Heisenberg), $SU(2)$ and $SU(1,1)$ groups which are the simplest examples of solvable, compact and non compact respectively, unitary Lie groups.

Therefore we now briefly summarize their defining commutation relations and their multiboson H.P. (Holstein-Primakoff [18]) realizations.

SU(2)

The commutation relations of $SU(2)$ are

$$[J_+, J_-] = 2J_3 \qquad [J_3, J_\pm] = \pm J_\pm \quad . \tag{2.8}$$

The UIR (unitary irreducible representation) corresponding to the eigenvalue $\sigma(\sigma+1)$ of the Casimir operator $J_3^2 + \frac{1}{2}(J_+J_- + J_-J_+)$ can be realized on a $2\sigma+1$ dimensional subspace of a fixed $\mathcal{F}_\lambda^{(k)}$ sector by means of the following generalized H.P. transformations:

$$J_+^{(k)} = (2\sigma + 1 - b_{(k)}^\dagger b_{(k)})^{\frac{1}{2}} b_{(k)}^\dagger = [J_-^{(k)}]^\dagger \quad ; \qquad J_3^{(k)} = b_{(k)}^\dagger b_{(k)} - \sigma \quad . \tag{2.9}$$

The special case $k = 1$ corresponding to the usual H.P. transformation has been considered too.

$SU(1,1)$

The commutation relations of $SU(1,1)$ are

$$[K_+, K_-] = -2K_3 \qquad [K_3, K_\pm] = \pm K_\pm \quad . \tag{2.10}$$

The UIR representation corresponding to the eigenvalue $\sigma(\sigma-1)$ of the Casimir operator $K_3^2 - \frac{1}{2}(K_+K_- + K_-K_+)$ is now infinite dimensional and can be realized on a whole fixed $\mathcal{F}_\lambda^{(k)}$ sector by means of the H.P. trasformations

$$K_+^{(k)} = (2\sigma - 1 + b_{(k)}^\dagger b_{(k)})^{\frac{1}{2}} b_{(k)}^\dagger = [K_-^{(k)}]^\dagger \quad ; \qquad K_3^{(k)} = b_{(k)}^\dagger b_{(k)} + \sigma \quad . \tag{2.11}$$

As for $SU(2)$ also the $k = 1$ case has been considered. For the $SU(1,1)$ case a bilinear realization of the algebra is also possible:

$$K_+ = \frac{1}{2} a^{\dagger 2} = [K_-]^\dagger \quad ; \qquad K_3 = \frac{1}{4}(2a^\dagger a + 1) \quad . \tag{2.12}$$

There are two UIR acting on the $\mathcal{F}_0^{(2)}$ and $\mathcal{F}_1^{(2)}$ sectors of the $k = 2$ splitting of the Fock space, corresponding to $\sigma = \frac{1}{4}$ and $\sigma = \frac{3}{4}$ respectively. As we shall see in Sect.3.1, the usual Gaussian states [12],[13],[20],[21] are related to the $\sigma = \frac{1}{4}$ case.

Weyl-Heisenberg

The *W.H.* algebra is the algebra of particle operators (or equivalently of the position and momentum operators). The commutation relations are given in (2.1) and the UIR is the usual infinite dimensional Fock representation, realized on a whole fixed $\mathcal{F}_\lambda^{(k)}$ sector.

One can notice that both the $SU(2)$ and $SU(1,1)$ H.P. realizations reduce to the *W.H.* algebra in the limit $\sigma \to \infty$ (in Sect.3.1 we shall give an intuitive geometrical interpretation of such limit).

In conclusion we recall that one can relize both the $SU(2)$ and $SU(1,1)$ UIR using bilinear products of bose operators corresponding to more than one oscillator mode (see for example ref. [22]).

3. The new states, their distributions and moments

3.1 Definition of the states

We focus our analysis on states $|\omega\rangle$ corresponding to zero-average position and momentum, since such an average can be arbitrarily changed to any desired value by a simple traslation

$$|z\rangle_\omega = D(z)|\omega\rangle \quad , \tag{3.1}$$

where $D(z)$ is the unitary displacement operator:

$$D(z) = \exp(za^\dagger - z^*a) \quad . \tag{3.2}$$

$a^\dagger$ and a denote the usual creation and annihilation operators $[a, a^\dagger] = 1$. In fact, for, say, the position $\hat{q} = \frac{1}{\sqrt{2}}(a + a^\dagger)$, one has

$${}_\omega\langle z|\hat{q}|z\rangle_\omega = \sqrt{2}Rez \quad . \tag{3.3a}$$

Thus, the generic nth moment is given by

$${}_\omega\langle z|(\hat{q} - \langle\hat{q}\rangle)^n|z\rangle_\omega = \langle\omega|\hat{q}^n|\omega\rangle \equiv \chi_\omega^{(n)} \quad . \tag{3.3b}$$

Analogous result hold for the momentum operator $\hat{p} = \frac{i}{\sqrt{2}}(a - a^\dagger)$.

The property that $|\omega\rangle$ is a zero average state is guaranteed if one assumes

$$|\omega\rangle = \hat{S}_\omega|0\rangle \quad , \tag{3.4}$$

where $\hat{S}_\omega$ is a unitary squeezing operator, which is an analytic function of multiparticle operators.

Furthermore, in view of the comparison we are interested in, between squeezed states and the customary coherent states (which have a Gaussian distribution for the canonical variables), we construct even distributions using an even number of particle creators. The usual squeezing operator [12],[20], which gives rise to a gaussian distribution,

$$\hat{S}(\zeta)_{Gauss} = \exp\left[\frac{1}{2}(\zeta a^{\dagger 2} - \zeta^* a^2)\right] \quad , \tag{3.5}$$

satisfies both the above requirements. Fisher, Nieto, and Sandberg [11] have proposed generalizations of the operator (3.5) in the form

$$\hat{S}_{(k)}(\zeta) = \exp(\zeta a^{\dagger k} - \zeta^* a^k + h_k) \tag{3.6}$$

where $h_k = h_k^\dagger$ is a polynomial in a and $a^\dagger$ with powers up to $(k-1)$. The resulting squeezed states cannot be treated in general by analytic methods. Indeed, for example, the Taylor expansion of the vacuum expectation value $\langle 0|\hat{S}_k|0\rangle$ leads to a series with zero radius of convergence (see also ref. [1]) and numerical computations could be performed only resorting to Padé approximants [14]. Only very special cases of operators of the form (3.6) can be analytically handled, i.e., for example, when $\hat{S}_k$ is the evolution operator corresponding to an hamiltonian which is a power of a bilinear operator [23].

One can easily understand the appearence of the formal analytical divergencies induced by the operator (3.6) trying to compute its action on the vacuum vector. Doing that requires dealing with the B.C.H. (Backer-Campbell-Hausdorff) factorization of $\hat{S}_{(k)}(\zeta)$ in the form:

$$\hat{S}_{(k)}(\zeta) = \exp[f(\zeta)a^{\dagger k}]\exp(\mathcal{O}_0) \quad , \tag{3.7}$$

where $f(\zeta)$ is some suitable function of ζ and $\mathcal{O}_0$ is an operator which stabilizes the vacuum and gives only a normalization factor. Adopting this method one needs to compute, for example, iterated commutators of the form ($m \geq l$):

$$[a^l, a^{\dagger m}] = p(a^\dagger a)a^{\dagger l-m} \quad , \tag{3.8}$$

where $p(x)$ is a polynomial function. If $k > 2$ this procedure never ends, and infact explodes into an infinite dimensional Fock algebra which one is in general unable to handle. On the contrary if $k = 2$, as for the usual Gaussian squeezed states, the finite dimensional Lie algebra (2.12) is obtained, and the factorization (3.7) can be explicitly written. More precisely one can see that the usual Gaussian squeezed states are nothing but the generalized group theoretical coherent states of $SU(1,1)$ according to the general definition for an arbitrary Lie group given by Perelomov and Rasetti [15]. (Indeed it is well known that the usual harmonic-oscillator coherent states themselves are group theoretical coherent states for $W.H.$ group).

The last observation suggests that group theoretical coherent states are good candidates for a *non naive* generalization of the squeezed states. We recall their general definition.

The set of coherent states for a Lie group G is obtained using a UIR of the group, chosing a fixed vector $|\Omega\rangle$ in the representation space, and acting on it by the whole group. It turns out that the coherent states are labeled by means of the left cosets of the group G with respect to the soubgroup leaving $|\Omega\rangle$ invariant up to a phase factor. Resorting to the above definition we construct the generalized squeezed states for the $SU(2)$, $SU(1,1)$ and $W.H.$ groups using the H.P. realizations of Sect.2.2:

$$|\zeta; k, \sigma\rangle^{SU(2)} = \exp[\zeta J_+^{(k)} - \zeta^* J_-^{(k)}]|0\rangle \quad ; \tag{3.9a}$$

$$|\zeta; k, \sigma\rangle^{SU(1,1)} = \exp[\zeta K_+^{(k)} - \zeta^* K_-^{(k)}]|0\rangle \quad ; \tag{3.9b}$$

$$|\zeta; k\rangle^{W.H.} = \exp[\zeta b_{(k)}^\dagger - \zeta^* b_{(k)}]|0\rangle \quad ; \tag{3.9c}$$

$$|\zeta\rangle^{Gauss} = \exp[\frac{1}{2}(\zeta a^{\dagger 2} - \zeta^* a^2)]|0\rangle \quad . \tag{3.9d}$$

For the sake of completeness we have written in (3.9d) also the usual Gaussian squeezed states which correspond to the $\mathcal{D}^+(\frac{1}{4})$ discrete series UIR of SU(1,1) and not to the H.P. realization.

By the B.C.H. formula one can rewrite the states (3.9) in the more convenient form :

$$|\xi; k, \sigma\rangle^{SU(2)} = (1 + |\xi|^2)^{-\sigma} \exp[\xi J_+^{(k)}]|0\rangle \quad ; \tag{3.10a}$$

$$|\xi; k, \sigma\rangle^{SU(1,1)} = (1 - |\xi|^2)^{\sigma} \exp[\xi K_+^{(k)}]|0\rangle \quad ; \tag{3.10b}$$

$$|\xi; k\rangle^{W.H.} = e^{-\frac{1}{2}|\xi|^2} \exp[\xi b_{(k)}^\dagger]|0\rangle \quad ; \tag{3.10c}$$

$$|\xi\rangle^{Gauss} = (1 - |\xi|^2)^{\frac{1}{4}} \exp[\frac{1}{2}\xi a^{\dagger 2}]|0\rangle \quad . \tag{3.10d}$$

The relation between the parameter ξ labeling the states in (3.10) and the parameter ζ in (3.9) is reported in the first column of Table I.

In the Fock basis the squeezed states read as follows:

$$|\xi; k, \sigma\rangle^{SU(2)} = (1 + |\xi|^2)^{-\sigma} \sum_{n=0}^{\infty} \binom{2\sigma}{n}^{\frac{1}{2}} \xi^n |kn\rangle \quad ; \tag{3.11a}$$

$$|\xi; k, \sigma\rangle^{SU(1,1)} = (1 - |\xi|^2)^{\sigma} \sum_{n=0}^{\infty} \binom{2\sigma + n - 1}{n}^{\frac{1}{2}} \xi^n |kn\rangle \quad ; \tag{3.11b}$$

$$|\xi; k\rangle^{W.H.} = e^{-\frac{1}{2}|\xi|^2} \sum_{n=0}^{\infty} \frac{\xi^n}{\sqrt{n!}} |kn\rangle \quad ; \tag{3.11c}$$

$$|\xi\rangle^{Gauss} = (1 - |\xi|^2)^{\frac{1}{4}} \sum_{n=0}^{\infty} \binom{2n}{n}^{\frac{1}{2}} \left(\frac{1}{2}\xi\right)^n |2n\rangle \quad . \tag{3.11d}$$

Equations (3.11) manifestly show that the squeezed states thus constructed are indeed multiphoton states.

From eqs. (3.11) the probability distribution of the number operator is easily obtained:

$$\mathcal{N}^{SU(2)}_{(\xi,k,\sigma)}(kn) = (1 + |\xi|^2)^{-2\sigma} \binom{2\sigma}{n} |\xi|^{2n} \quad ; \tag{3.12a}$$

$$\mathcal{N}^{SU(1,1)}_{(\xi,k,\sigma)}(kn) = (1 - |\xi|^2)^{2\sigma} \binom{2\sigma + n - 1}{n} |\xi|^{2n} \quad ; \tag{3.12b}$$

$$\mathcal{N}^{W.H.}_{(\xi,k)}(kn) = e^{-|\xi|^2} \frac{|\xi|^{2n}}{n!} \quad ; \tag{3.12c}$$

$$\mathcal{N}^{Gauss}_{\xi}(2n) = (1 - |\xi|^2)^{\frac{1}{2}} \binom{2n}{n} \left(\frac{|\xi|}{2}\right)^{2n} \quad ; \tag{3.12d}$$

$$\mathcal{N}^{SU(2)}_{(\xi,k,\sigma)}(p) = \mathcal{N}^{SU(1,1)}_{(\xi,k,\sigma)}(p) = \mathcal{N}^{W.H.}_{(\xi,k)}(p) = 0 \quad ; \qquad p \neq kn \tag{3.12e}$$

$$\mathcal{N}^{Gauss}_{\xi}(2n+1) = 0 \quad . \tag{3.12f}$$

Equations (3.12a)-(3.12e) represent, respectively, the binomial, negative-binomial and Poisson distributions in the many-photon variable kn.

We want now to compare the statistical properties inherent in the different states. The squeezing parameter ξ introduced in eqs.(3.10) does not lend itself to a trasparent physical interpretation and appears therefore somewhat ambiguous. On the other hand, one can see from Table I that the quantity describing the number fluctuations,

$$\delta = \frac{\Delta n}{\langle n\rangle} \quad , \tag{3.13}$$

of the number operator is inversely proportional to $|\xi|$ with a coefficient depending only on the group representation. We adopt therefore δ^{-1} as a good independent variable to compare the squeezing properties of the different states.

3.2 Squeezing properties of the new states.

In this section we give a detailed analysis of the second moment of the position variable $\hat{q}$ normalized to the value corresponding to the vacuum state

$$\chi^{(2)} = \frac{\langle\omega|\hat{q}^2|\omega\rangle - \langle\omega|\hat{q}|\omega\rangle^2}{\langle 0|\hat{q}^2|0\rangle - \langle 0|\hat{q}|0\rangle^2} \quad , \tag{3.14}$$

by varying the state $|\omega\rangle$ in the set (3.11) and in the direction of the maximal squeezing, i.e. for negative real ξ.

Figure 1 reports the second moments $\chi^{(2)}$ for various two-photon squeezed states as functions of δ^{-1} (the latter two in the $\sigma = 3$ representation). For the sake of comparison, the results for Gaussian states are also shown. One can notice that among all states the Gaussian ones exibit the best squeezing for a fixed value of δ^{-1}. However, they cannot attain a fluctuation in the observable number of photons lower than $\sqrt{2}$;in other words, the Gaussian states are *photon noisy*. Furthermore, as $\chi^{(2)}_{Gauss}$ is a monotonic decreasing function of δ^{-1} , the best squeezing corresponds to the lowest $\hat{n}$ fluctuation. On the oter hand, all the other non-Gaussian states give rise to functions $\chi^{(2)}(\delta^{-1})$ which are not monotonic but exibit a local minimum. Among them only the $SU(1,1)$ states can be completely squeezed ($\chi^{(2)} = 0$).

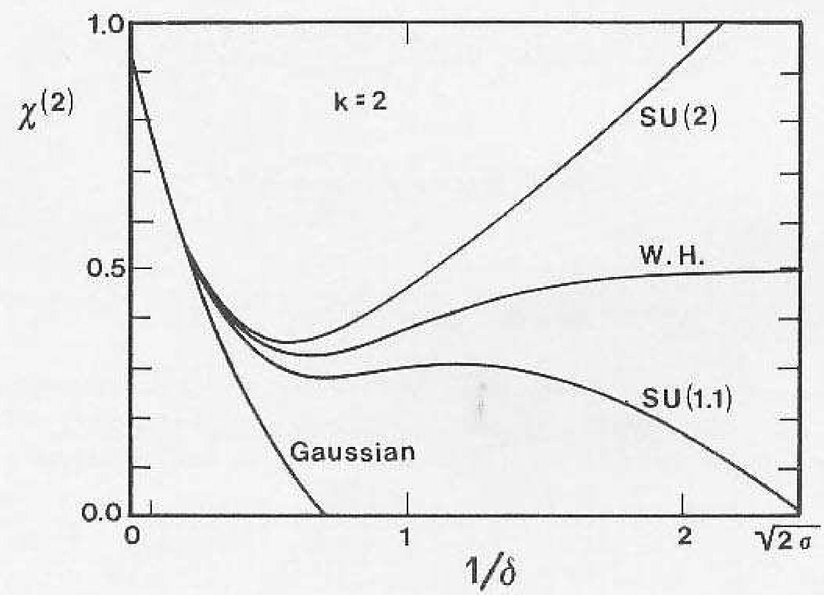

Fig.1 - Squeezing (i.e., second moment for negative squeezing parameter) vs the inverse $\hat{n}$ fluctuation parameter δ^{-1}, for various two-photon squeezed states; $SU(2)$ and $SU(1,1)$ states correspond to the $\sigma = 3$ representation (from ref.[7]).

One can notice as well that, in general, non-Gaussian states can attain a photon-number fluctuation smaller than those of the Gaussian states. In particular, the $W.H.$ states can have an arbitrarily small photon noise, but they are *squeezed limited* in that the second moment $\chi^{(2)}_{W.H.}$ exibits an *absolute* minimum $\chi^{(2)}_{W.H.,min} = 0.31744$ corresponding to $\delta^{-1} = 0.64675$. The $SU(1,1)$ states can be squeezed to zero second moment in correspondence to the optimal value $\delta^{-1} = \sqrt{2}\sigma$. Therefore one can simultaneously reduce to zero both $\hat{n}$-noise and $\hat{q}$-noise in the limit $\sigma \to \infty$. It is worth pointing out that whereas for $W.H.$ states the local minimum is also a global one, for the $SU(1,1)$ states the absolute minimum is zero (numerical values of relative minima for large σ are given in Refs.[7] and [4]). Finally, the $SU(2)$ states are no longer squeezed ($\chi^{(2)}_{SU(2)} > 1$) for small $\hat{n}$ fluctuations.

Figure 2 shows the reduced absolute fluctuations $(\Delta n/k)^2$ vs δ^{-1} for the same states considered in Fig. 1. It appears from this figure, comparing it with the previous one, that the better the squeezing the higher the photon-number fluctuations. In particular, the Gaussian states exibit the highest photon noise.

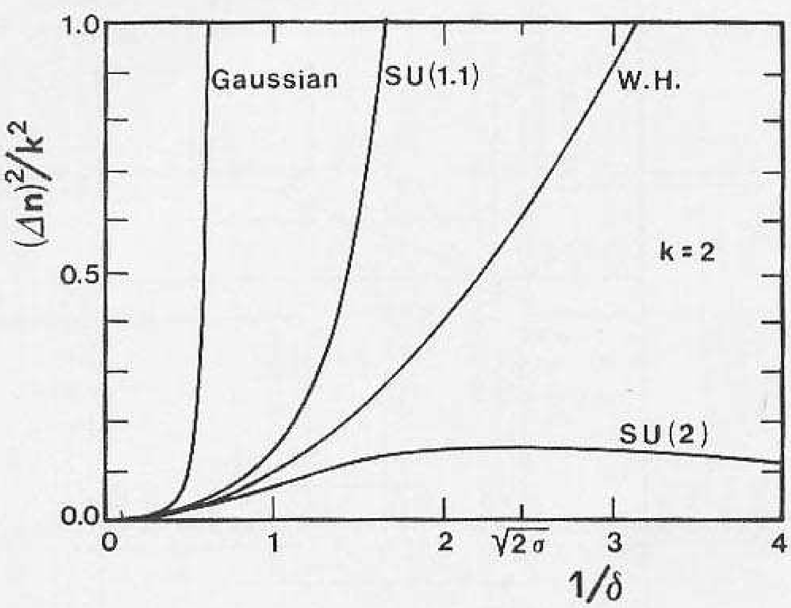

Fig.2 - Reduced $\hat{n}$ square variance vs the inverse $\hat{n}$-fluctuation parameter δ^{-1} for the same two-photon squeezed states of Fig. 1 (from ref.[7]).

In the limit of squeezing to zero second moment, $\delta \to \sqrt{2}$ for the Gaussian states or $\delta \to 1/\sqrt{2\sigma}$ for the $SU(1,1)$ states, the $\hat{n}$-variance increases asymptotically to infinity for both states. $(\Delta n/k)^2_{W.H.}$ grows parabolically with δ^{-1}, whereas $(\Delta n/k)^2_{SU(2)}$ shows a maximum, and decreases to zero as δ^{-1} tends to infinity, as $\sim (\delta^{-1})^{-2}$.

From Figs. 1 and 2, one can then conclude that the local minimum of $\chi^{(2)}$ for the non-Gaussian states can be considered as an optimum situation as it provides the best compromise between the requisite of maximum squeezing and that of minimum absolute noise in the photon number.

3.3 Probability distributions

In this section we show some numerical results (ref.[5]) concerning the position probability distribution

$$\mathcal{Q}_\omega(q) = |\langle q|\omega\rangle|^2 \quad , \tag{3.15}$$

and the number distribution

$$\mathcal{N}_\omega(n) = |\langle n|\omega\rangle|^2 \quad , \tag{3.16}$$

for some states in the set (3.11).

Figures 3(a)-3(h) represent $\mathcal{Q}_\omega(q)$ for the $W.H.$ states for various choices of k and for different values of $\zeta = \rho e^{i\phi}$ (the probability distributions for the $SU(2)$ and $SU(1,1)$ states are analogous).

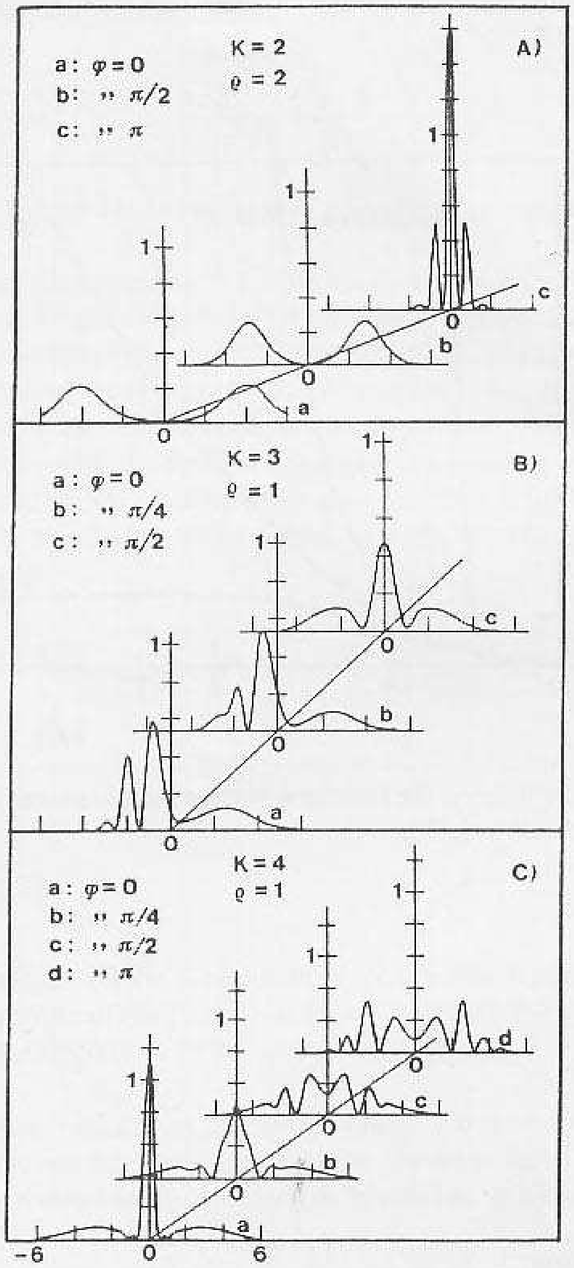

Fig.3 - The probability distribution function $\mathcal{Q}_{(k,\zeta}$ vs the (dimensionless) position q, for different values of k and different choices of $\zeta = \rho e^{i\phi}$ (from ref.[5]).

One notices how such distributions exhibit a sensible deviation from the Gaussian behavior. The functions corresponding to even k are symmetric under the exchange $q \to -q$, whereas there is no symmetry for odd k except for $\phi = \pi/2$. A characteristic feature of the functions $\mathcal{Q}_{(k,\zeta)}(q)$ is that they show an increasing number of zeros when ρ is increased at fixed ϕ, for any k. The same effect, i.e., a richer structure corresponding to a larger number of nodes, appears when k is increased keeping ζ fixed.

As for the moments of these probability distributions, a general theorem is proven in ref. [8] concerning the probability distribution of every multiphoton state (i.e. a state which is a surposition of eigenstates $|kn\rangle$ with varying n and fixed k):

For a k-photon state $|\omega\rangle$, only the moments $\chi_\omega^{(2N)}$ corresponding to $2N \geq k$ can be squeezed for even k, $N \geq k$ for odd k .

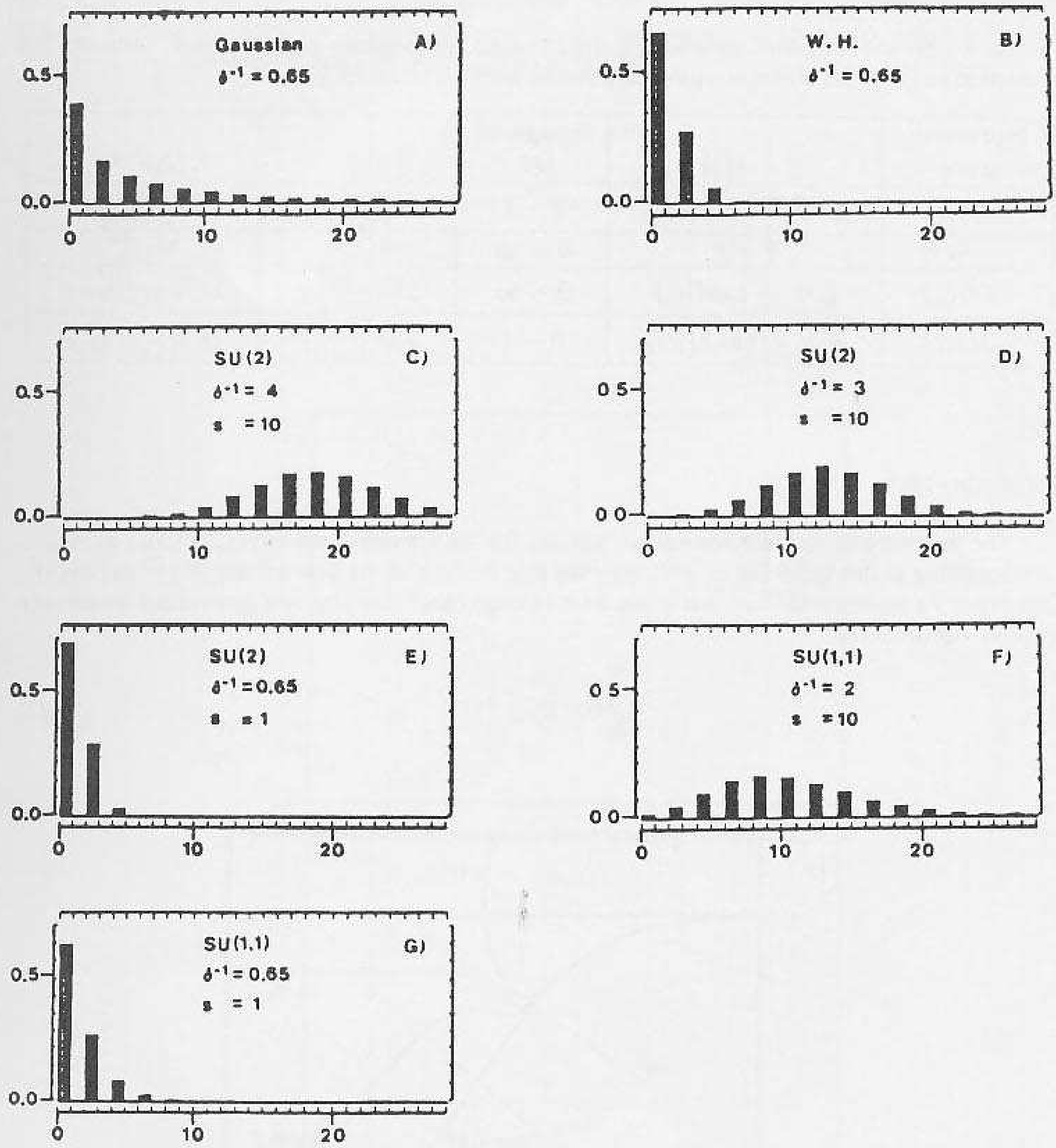

Fig.4 - The probability distribution function $\mathcal{N}(n)$ of the $\hat{n}$ operator for the $k = 2$ states for different values of σ and $|\xi|$.

Number probability distribution

Figure 4 shows some $\mathcal{N}_\omega(n)$ distributions for all the different types of states and a few values of

σ (for the $SU(2)$ and $SU(1,1)$ cases). Notice how the negative binomial distribution appears as the slowest decaying one for large n: as $|\xi|^2$ and/or σ are increased, one obtains a maximum for larger and larger n and more and more peacked functions, getting as a result smaller δ .

Restricting our attention to the Poissonian sub-Poissonian shape of the distributions, one can easily check from Table I that the only sub-Poissonian distribution is the binomial distribution, related to $SU(2)$ states: all other distributions are super-Poissonian (included that of the Gaussian squeezed vacuum).

TABLE 1 - Backer-Hausdorff parameter ξ and its range and photon-number average, variance, and fluctuation vs ξ, for the different squeezed states considered (from ref.[7]).

Squeezed state	$\xi = \xi(\varsigma)$	Range of $\lvert\xi\rvert$	$\langle n \rangle$	$\langle \Delta n \rangle^2$
Gauss	$\xi = \frac{\varsigma}{\lvert\varsigma\rvert}\tanh(\lvert\varsigma\rvert)$	$0-1$	$\frac{\lvert\xi\rvert^2}{1-\lvert\xi\rvert^2}$	$2\frac{\lvert\xi\rvert^2}{(1-\lvert\xi\rvert^2)^2}$
WH	$\xi = \varsigma$	$0-\infty$	$k\lvert\xi\rvert^2$	$k^2\lvert\xi\rvert^2$
SU(2)	$\xi = \frac{\varsigma}{\lvert\varsigma\rvert}\tan(\lvert\varsigma\rvert)$	$0-\infty$	$2k\sigma\frac{\lvert\xi\rvert^2}{1+\lvert\xi\rvert^2}$	$2k^2\sigma\frac{\lvert\xi\rvert^2}{(1+\lvert\xi\rvert^2)^2}$
SU(1,1)	$\xi = \frac{\varsigma}{\lvert\varsigma\rvert}\tanh(\lvert\varsigma\rvert)$	$0-1$	$2k\sigma\frac{\lvert\xi\rvert^2}{1-\lvert\xi\rvert^2}$	$2k^2\sigma\frac{\lvert\xi\rvert^2}{(1-\lvert\xi\rvert^2)^2}$

3.4 Scaling laws

The existence of the two vertical asymptotes for the Gaussian and $SU(1,1)$ states in Fig. 2, corresponding to the vanishing of $\chi^{(2)}$, suggests that we look at the dependence of $\chi^{(2)}$ vs $(\Delta n/k)^2$. One expects a scaling relation – that in the limit of large $(\Delta n)^2$ should give a generalized uncertainty relation – in the form

$$\chi^{(2)} \sim (\Delta n)^{-2\gamma} \tag{3.17}$$

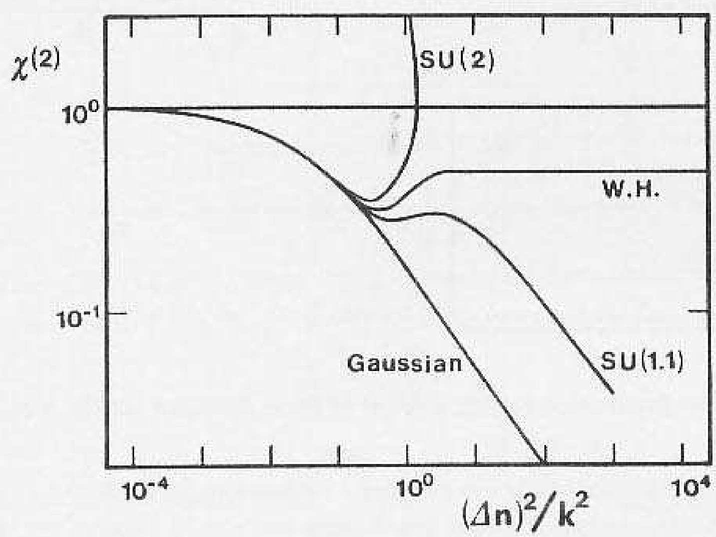

Fig.5 - Log-log plot of squeezing vs reduced absolute $\hat{n}$ fluctuation for the same squeezed states of Figs. 1 and 2 (from ref.[7]).

Figure 5 shows the log-log plot of squeezing versus reduced absolute photon-number fluctuation for all the two- photon states of Fig. 1. One can notice how $\gamma = \frac{1}{2}$ for both the Gaussian and the $SU(1,1)$ states; in the latter case γ being independent of the value of σ, provided it is finite. The proportinality constant depends on both the state [Gaussian or $SU(1,1)$] and on the representation [σ]. Thus the parameter γ can be thought of as a *universal* scale exponent. One should observe that, considering the $W.H.$ states as the $\sigma \rightarrow \infty$ limit of $SU(1,1)$, the universal behavior is broken in the same limit, and we have $\gamma = 0$.

Scaling laws analogous to (3.17) can be found for higher moments as well. Somewhat unexpectedly, scaling laws for second- and higher-order moments appear as well for all the states corresponding to the local minima of the moments themselves versus δ^{-1}.In this case, the parameter whereby the two uncertainties $\chi^{(2N)}$ and $(\Delta n)^2$ can be connected is the representation label σ, which is the only remaining free variable. Generalized scaling laws of the form

$$\chi^{(2N)}_{(k)} \sim (\Delta n)^{-2\gamma_k(N)} \tag{3.18}$$

can be obtained by eliminating σ^{-1} between $\chi^{(2N)}$ and $(\Delta n)^2$.

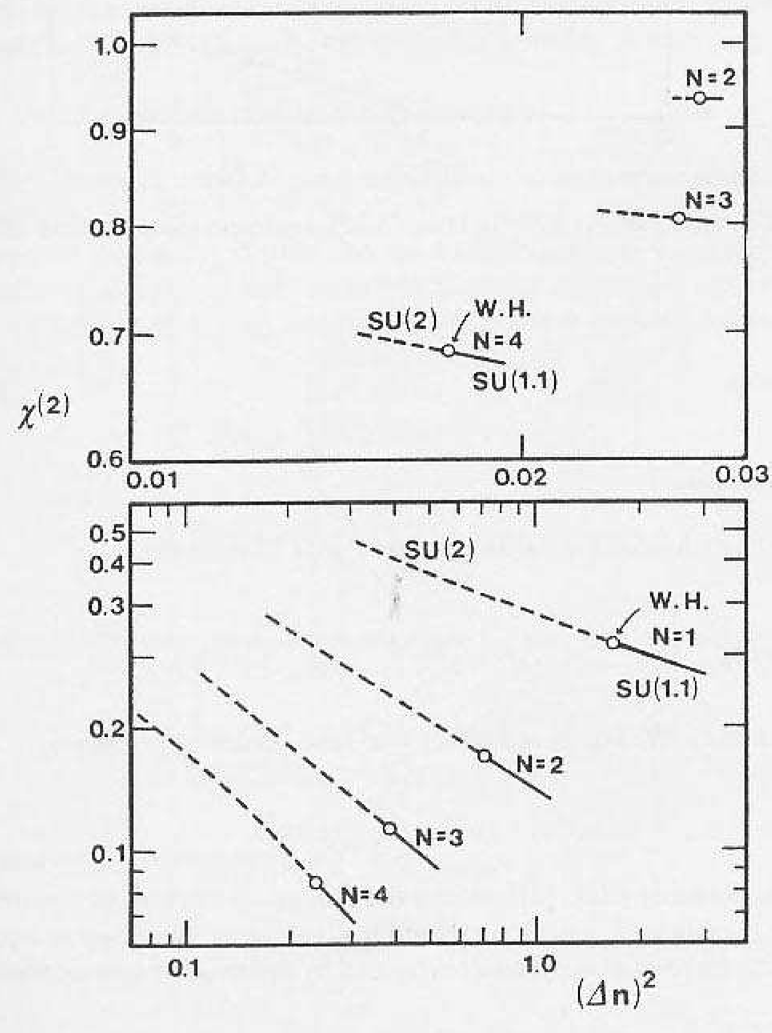

Fig.6 - Generalized squeezing for the $2N$th moments vs absolute $\hat{n}$ fluctuation at the local minimum (log-log plot): (a) $k = 4$, $N = 2 \div 4$; (b) $k = 2$, $N = 1 \div 4$ (from ref.[7]).

Figure 6 shows the log-log plots of the optimal moments $\chi^{(2N)}_{min}$ versus the corresponding $\hat{n}$-variance, which manifestly exhibit a power-law behavior of the form (3.18). It is interesting to point out how, in this representation, all states $[W.H., SU(2)$, and $SU(1,1)]$ lie on the same straight lines. The exponents $\gamma_k(N)$ are positive numbers less than 1, whose dependence on N and k is shown in Fig. 7. Notice that $\gamma_k(N)$ is monotonically increasing with N and decreasing with k, (on the contrary one obtains that the proportionality constant is decreasing with N and increasing with k).

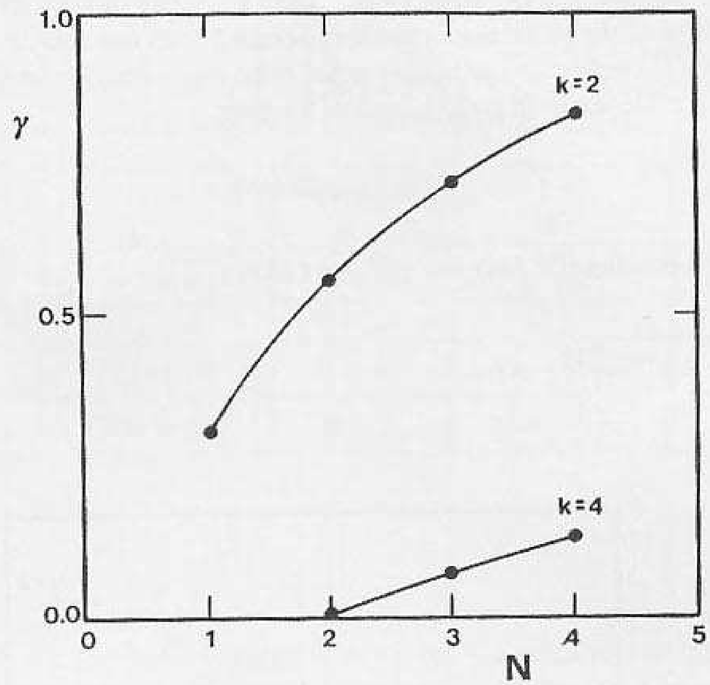

Fig.7 - The exponent $\gamma_k(N)$ of the scaling laws (3.18), corresponding to plots of Fig. 6, vs N for $k = 2, 4$ (from ref.[7]).

4. Fractional photons

4.1 The algebraic definition

In Sect 2.1 we have defined the transformation $F_{(k)}$ of (2.3),(2.4):

$$F_{(k)} : a^\dagger \to F_{(k)}(a^\dagger) = b^\dagger_{(k)} \quad , \quad F_{(k)}(a) \equiv [F_{(k)}(a^\dagger)]^\dagger \quad , \tag{4.1}$$

for positive integer k only. We note that $F_{(1)}(a^\dagger) = a^\dagger$, and, with a little algebra,

$$F_{(k)} \circ F_{(k')}(a^\dagger) = F_{(kk')}(a^\dagger) \quad .$$

We may at least formally (Ref. [8]), extend the semigroup of nonlinear trasformations (4.1) to the Abelian group $\{F_{(k)} \text{rational} k > 0\}$ (one should really think of this group as a group of canonical transformations acting on pairs of conjugate operators), by defining the inverse trasformation $F^{-1}_{(k)}$ by

$$F_{(k)}{}^{-1} \circ F_{(k)}(a^\dagger) = a^\dagger = F_{(1)}(a^\dagger) \quad .$$

We may therefore equate $F^{-1}_{(k)} = F_{(1/k)}$, whence

$$F_{(k)}^{-1} \circ F_{(k')} = F_{(k'/k)} = F_{(r)}$$

where $r \equiv k'/k$ is a positive rational number. It is this extension which allows us to define the notion of *fractional photons*.

The above formal structure is equivalent to considering the action of k'-boson operators on a $\mathcal{F}$ sector with $k \neq k'$. Focusing our attention on the particular sector $\mathcal{F}_0^{(k)}$, the k'-boson action is given by the following matrix elements:

$$\langle km|(b^\dagger_{(k')})^u (b_{(k')})^v|km'\rangle = \left(\frac{[[km/k']]![[km'/k']]!}{[[km/k'-u]]![[km'/k'-v]]!}\right)^{\frac{1}{2}} \delta_{m,m'+s} \quad ,$$

where we have defined s by $s = (k'/k)(u-v)$. When $u = v$, then $s = 0$ and the expectation (2.3) always has nonzero values (for $m = m'$). When $u \neq v$, the expression (2.3) vanishes unless s is an integer; that is, $(k'/k)(u-v)$ is an integer.

Note that expression (2.3) depends only on k' and k through their ratio: $r = k'/k$. Here r is the positive rational fraction of the fractional trasformation $F_{(r)}$. We may equate expression (2.1) *formally* to an expectation involving fractional photons:

$$\langle km|(b^\dagger_{(k')})^u (b_{(k')})^v|km'\rangle = \langle m|(b^\dagger_{(r)})^u (b_{(r)})^v|m'\rangle$$

Thus the claim is not that such fractional photon modes really exist, but that physical experiments involving integral numbers of photons can be interpreted as behaving in such fractional mode.

4.2 Physical states, their distributions and squeezing properties.

In ref.[10] physical quantum states are constructed which have the same probability distributions of fractional photon states.

The definition of the probability distribution for fractional photon states is based on the construction of a complete set of eigenvectors for the two mutually commuting operators $\hat{Q}_{(k)}$ and $\hat{D}_{(k)}$ defined in Sect.2.1. The diagonalization procedure is standard, and gives the following result:

$$|Q,\lambda\rangle_{(k)} = \sum_{l=0}^{\infty} C_l(Q)|lk+\lambda\rangle \quad , \qquad C_l(Q) = \frac{e^{\frac{Q^2}{2}} H_l(Q)}{\sqrt{2^l l! \sqrt{\pi}}} \quad , \tag{4.2}$$

where $H_l(Q)$ are the usual Hermite polynomials of degree l. One can easily check that:

$$\hat{Q}_{(k)}|Q,\lambda\rangle_{(k)} = Q|Q,\lambda\rangle_{(k)} \quad ; \qquad D_{(k)}|Q,\lambda\rangle_{(k)} = \lambda|Q,\lambda\rangle_{(k)} \quad . \tag{4.3}$$

If one considers next a k-photon state in the $\mathcal{F}_0^{(k)}$ sector:

$$|\omega\rangle_{(k)} = \sum_{m=0}^{\infty} \omega_m |km\rangle \quad , \tag{4.4}$$

one can construct the probability distribution of the canonical variable $\hat{Q}_{(k')}$ for the k-photon state $|\omega\rangle_{(k)}$ $k \neq k'$ as

$$\mathcal{P}_\omega^{(t)} = \sum_{\lambda=0}^{k'-1} |{}_{(k')}\langle Q, \lambda||\omega\rangle_{(k)}|^2 = \sum_{l,m=0}^{\infty} \omega_l^* \omega_m C_{[[tl]]}(Q) C_{[[tm]]}(Q) \delta_{\langle\langle tl \rangle\rangle \langle\langle tm \rangle\rangle} \quad , \tag{4.5}$$

where $\langle\langle x \rangle\rangle = x - [[x]]$ denotes the fractional part of x. Eq. (4.5) shows clearly that the probability distribution depends only on $t = 1/r = k/k'$ and can thus be referred to as fractional photon probability distribution.

As an example we select, as k-photon vector $|\omega\rangle_{(k)}$, one among the generalized k-photon states of eq.(3.11), namely the $W.H.$ coherent state:

$$\begin{aligned} |\omega\rangle_{(k)} &\equiv |\xi; k\rangle^{W.H.} \quad , \\ \omega_n &= \frac{e^{-\frac{1}{2}|\xi|^2} \omega^n}{\sqrt{n!}} \quad , \end{aligned} \tag{4.6}$$

and restrict our attention to the special case $t = 1/n$. In this case the probability distributions are almost gaussian (see for example Fig. 8). More precisely they approach the vacuum gaussian shape for $t \to 0$ (and obviously for $|\omega|^2 \to 0$) in agreement with an intuitive physical meaning of vacuum as zero-fraction photon state. On the other hand if one increases $|\omega|^2$ at fixed t, the gaussian shape changes to a richer structure, correspond ing to a larger number of local maxima and minima. For very large $|\omega|^2$ the maxima raise up more and more sharply around the average value, and in the limit $|\omega|^2 \to \infty$ the distribution merges into a generalized function, as for the usual integer-k multi-photon distributions.

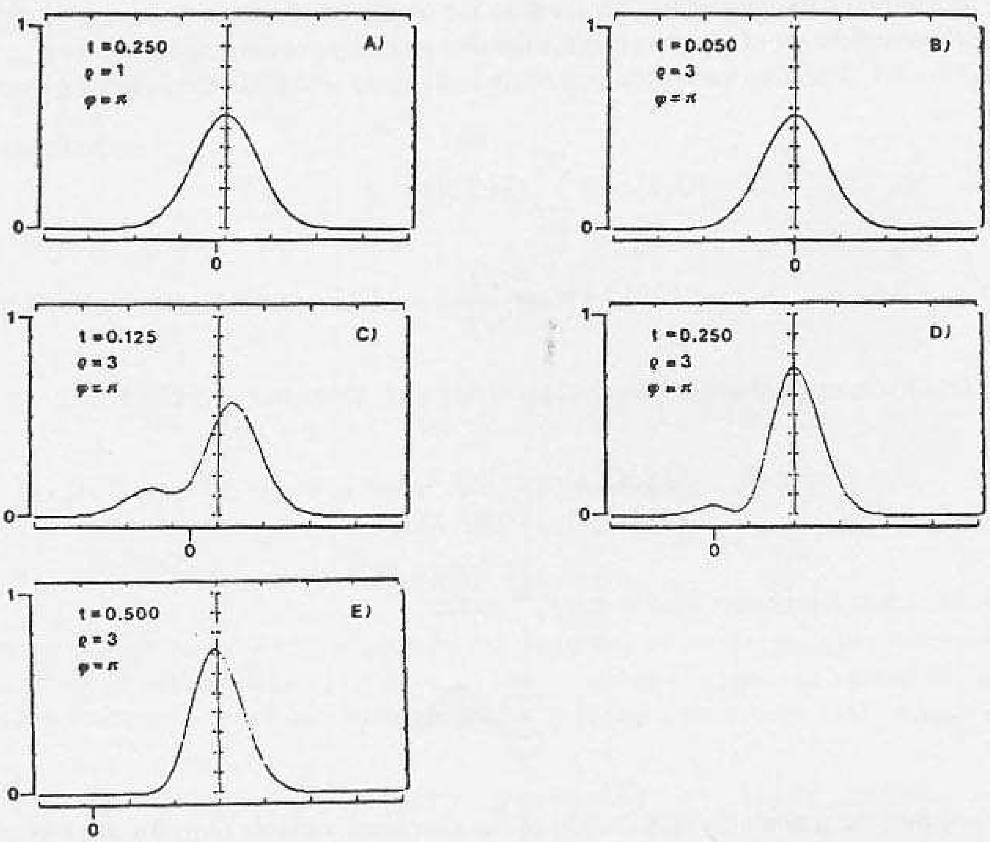

Fig.8 - The probability distribution $\mathcal{P}_\omega^{(t)}$ vs the canonical variable Q for the $W.H.$ fractional states, for some values of $\rho = |\omega|$ and $t = 1/n$.

Squeezing is obtained increasing $|\omega|$ along the negative real direction as usual. In Fig. 9 the second moment for the canonical variable $\hat{Q}_{(k)}$ is plotted versus $\rho = |\omega|$ for various values of $t = 1/n$. One can see that better squeezing is obtained for larger ρ and smaller t. One can check [10] that for increasing values of ρ the squeezing asymptotically approaches the constant value $\chi^{(2)} \sim t$. One gets thus the nice notion that *fractioning photons is equivalent to squeezing photon distributions.*

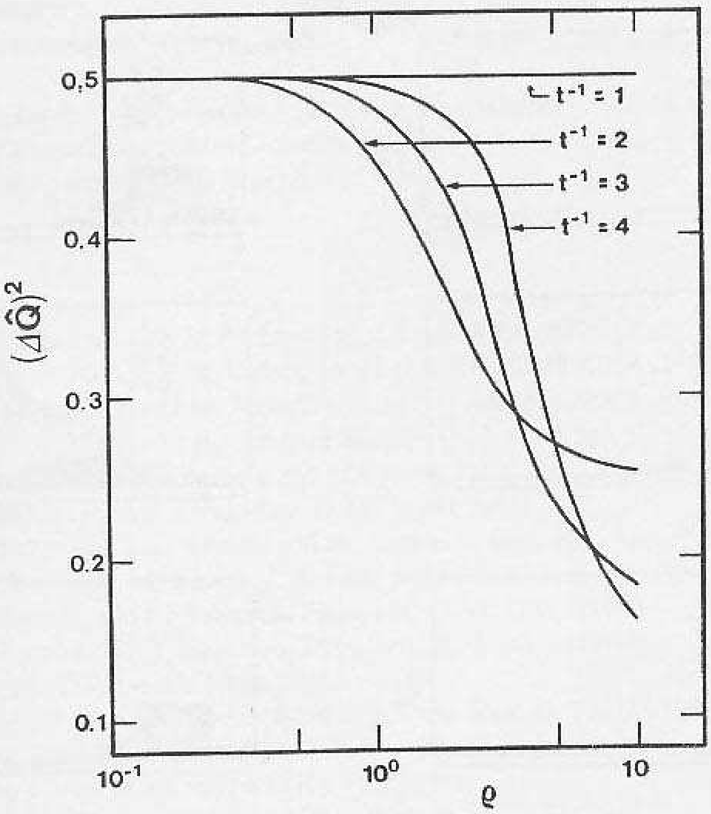

Fig.9 - Dependence of the second moment $\chi^{(2)}$ on the squeezing parameter $\rho = |\omega|$ for the *W.H.* fractional states for various values of $n = 1/t$ (from ref.[8]).

The probability distribution considered thus far refers to the canonical variable $\hat{Q}_{(k)}$ which does not have a simple physical meaning. One can inquire whether it is possible to construct a physical Fock state which has exactly both the probability distribution $\mathcal{P}_{\omega}^{(t)}$ and the number distribution $\mathcal{N}_{\omega}^{(t)}(N)$

$$\mathcal{N}_{\omega}^{(t)}(N) = \sum_{\lambda=0}^{h-1} |{}_{(h)}\langle N, \lambda|\omega\rangle_{(k)}|^2 = e^{-|\omega|^2} \sum_{\lambda=0}^{p-1} \frac{|\omega|^{2(pn+\lambda)}}{(pl+\lambda)!} \quad , \tag{4.7}$$

but referred to the usual position $\hat{q}$ and number $\hat{n}$ variables. The answer is positive: the physical fractional photon state is a mixed state defined by a density matrix (ref.[10]); i.e. the fractional photon is essentially a statistical object.

One can understand the physical features of the fractional states by looking at their number probability distribution (4.7). In Fig. 10 a few probability distributions are plotted for different t and various values of $\rho = |\omega|$ (for the sake of comparison, the usual coherent state correspondig to $t = 1$ is included as well). One can see how the fractional state has decreasing mean number of photons for decreasing t (it is straightforward to compute that, for large ρ, $\langle n\rangle \sim t$. Furthermore as ρ is increased a sub-Poissonian distribution is obtained (indeed one can analytically check that $(\Delta n)^2/\langle n\rangle \sim t$ for large ρ whereas $(\Delta n)^2/\langle n\rangle \sim \frac{1}{\rho^2}$.

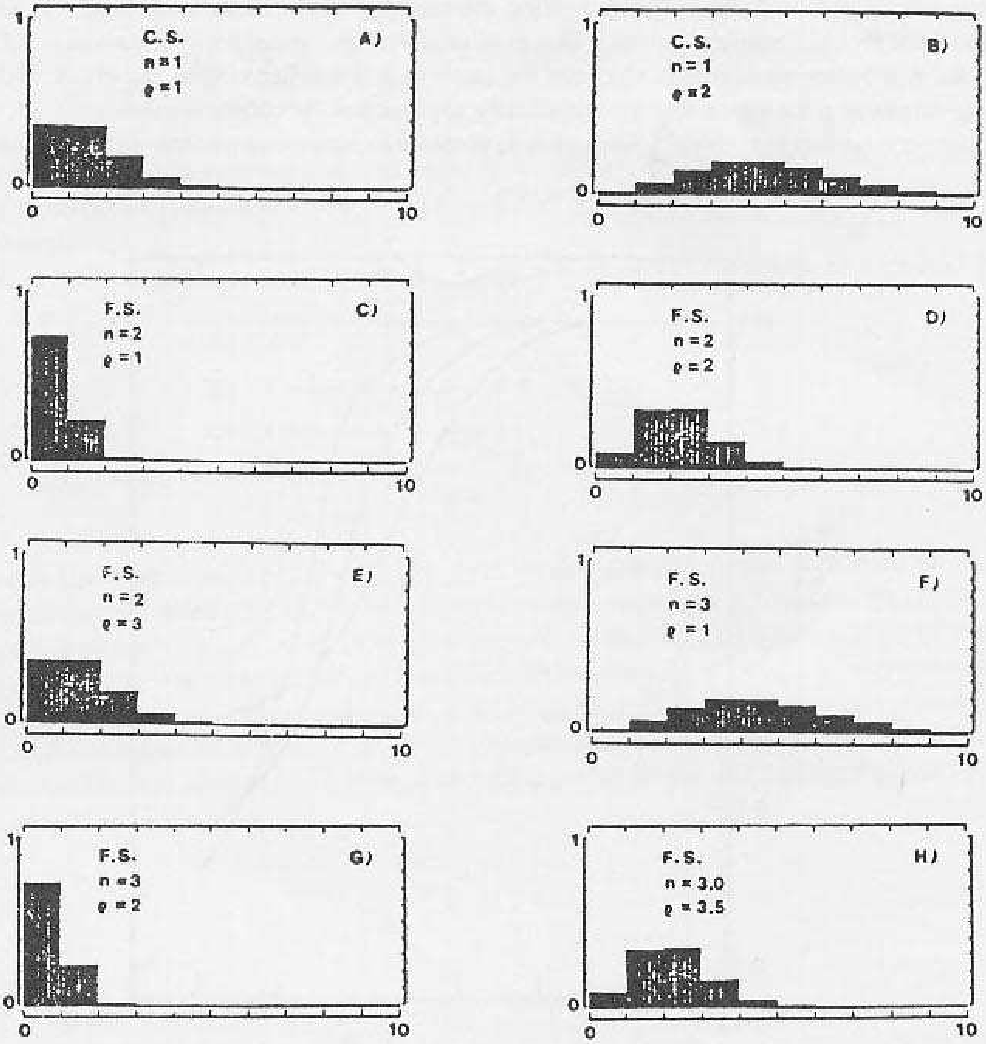

Fig.10 - Number probability distributions for the $W.H.$ fractional states for some values of $n = 1/t$ and $\rho = |\omega|$.

In conclusion, the most intriguing physical feature of the fractional photon state is that *one can simultaneously obtain complete squeezing, sub-Poissonian number distribution and very small number fluctuations* taking the limit $\rho \to \infty$ followed by the limit $t \to 0$.

5. Conclusions

We conclude with two comments related mainly with the problem of physically realizing the (formal) states described in this paper.

The information most relevant for the realization of a specified quantum state is of course the interaction whereby the state itself is generated as a pure states, stemming out of the appropriate vacuum. In all the cases described in this paper the resulting interaction – one can think of an Hamiltonian, roughly proportional to the logarithm of the squeezing operator – is very complicated in the regular phase space (it is in general described by an infinite power series of the single photon creation and annihilation operators), and typically, when expressed in terms of p and q variables does not show manifestly the structure of kinetic plus anharmonic potential energy one should expect. However, preliminary numerical analysis has shown that at least locally this in in fact the case (the potential exhibiting a characteristic double well shape). On the other hand the Hamiltonian should have the algebra corresponding to the state considered as dynamical (spectrum generating) algebra; namely there should exist non-manifest symmetries of the dynamical system resulting in the property that the Hamiltonian is generated by commutation relation in a finite rank algebraic structure. This hints to the existence of a set of action-angle variables in which the form of the interaction should be much simpler.

This is also supported by the feature – here shown explicitly for the fractional photon states – that very few single photon states are sufficient to realize the described squeezed states.

Further work along these lines is in progress.

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