A Quantum-Digital Universe

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Can Reality be simulated by a huge Quantum Computer? Do we believe that Reality is made of something more than interacting quantum systems? The idea that the whole Physics is ultimately a quantum computation—a strong quantum version of the Church-Turing hypothesis well synthesized by the Wheeler's coinage *It from bit*—is very appealing. It is theoretically very parsimonious: an Occam razor's quality-guaranteed description of the world. But, if this is the case, then we need to understand the entire Physics as emergent from the quantum computation. Here I will make a short exploration on how this may come about.

Ask a physicist what does he thinks the world is made of. Very likely the first impetuous answer will be: it is made of particles! But, at a second thought the answer will be: it is a Quantum Field. Particles are just *states* or the Field: they can be created and annihilated. We have indeed a beautiful Grand Unified Field Theory, and we are looking forward to see the Higg's boson at the LHC.

But what is the Quantum Field made of? Ultimately, it is made of quantum systems that are interacting, each system located at a different position in space. Things may be indeed more complicated than that, because the field is a *continuum*. But is Reality actually continuous? We don' t know: but it looks easier to think to Reality as a continuum. Now, suppose that this is not the case, namely Reality is ultimately discrete, and the continuum is only a mathematical fiction. Then, what else is out there more than interacting quantum systems? Is it space? No, space is a "nothingness". Is it Relativity? No, that's not a "thing": it is a way of looking at things. We thus come to the conclusion that Reality is made only of "interacting quantum systems", and this is precisely what we call a *quantum computer*. David Deutsch in his seminal paper *Quantum theory, the Church-Turing principle and the universal quantum computer* [1] rephrased the Church-Turing hypothesis as a truly physical principle. In short: every piece of physical reality can be perfectly simulated by a quantum computer. But now: what is the difference between Reality and its simulation? Its a matter for metaphysics: if Reality is indistinguishable from its simulation, then *it is* its simulation. The Universe is really a huge quantum computer: the computational universe of Seth Lloyd [2].

But we have more than that. Quantum Theory is ultimately a "theory of information", an idea that has been hanging around for many years [3-8] since the Wheeler's *It from bit* [9], and which has been also recently proved mathematically [10]. Therefore, if we adopt the Deutsch's Church-Turing principle, the notion of *Information* becomes the new big paradigm for Physics.

The scientific approach behind the computational paradigm. The *Informationalism* can be ultimately regarded as a new scientific approach, very close to the spirit of Niels Bohr and the Copenhagen school. Far from being speculative, the approach is truly *operational*, namely everything must be defined by a precise procedure—ultimately in terms of accurate quantum measurements. The real entities are the *events*, facts of the world describable by the basic language obeying the rules of predicate logic (the "facts" of Wittgenstein's *Tractatus*). Formulating a *Theory* of the observed (or potentially observable) events means building up a network of input-output connections between them. In a *causal theory* these connections are causal links: Quantum Theory is exactly a theory of this kind [10]. Translating these terms into computer-programming language, the events are the *subroutines*, and the causal links are the *registers* where information is written and read. Translating into physical terms: the links are the *systems* and the events are the *transformations*. The notion of "event" must be regarded as truly primordial: *events do not happen in space-time, they build-up space-time*. Stated in other words: space-time is our way of organizing events. The idea of deriving the geometry of space-time from a purely causal structure has been also hanging around for more than two decades after Raphael Sorkin opened the causal sets program [11].

Physicists often identify Theory with Reality, but Theory is only our way to connect phenomena, to relate an input with an output. Input and output are linked through cascades of *local interactions*, namely events that involve only a finite number of systems. In the quantum computer the subroutines (the potential events) are the unitary transformations of the *gates*, and the causal links are the *quantum systems*—the so-called *qubits*. In Fig. 1 a piece of quantum circuit is represented. The gate (box) performing the unitary transformation A reads information from two input registers (wires) which in turn are the output of two gates performing the unitary B, and so forth.

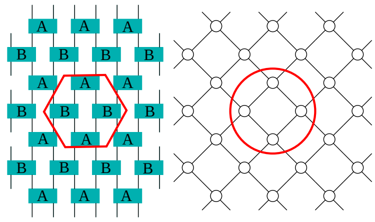

Figure 1. A portion of a quantum circuit (left) and its causal network representation (right). The hexagon (and the corresponding circle on the right) represents a tile which allows to recover the whole circuit upon translation. This is the equivalent of a Physical Law.

What is a Physical Law in this causal-network framework? It is a piece of network—a set of events (gates) along with causal links (wires) connecting them—by translating which we can build up the whole unbounded periodic network, corresponding to our supposition that the law is true everywhere and ever. Such representation of the physical law contains only its logical essence, stripped of the "conventional" part (e. g. the conventionality of simultaneity [12,13]).

The informational paradigm is a huge change of ontology: there is no *stuff* that supports the *qubits*, but stuff itself is made of qubits! This is a change of perspective that is hard to swallow. Those who strongly believe in the reality of space-time with "objects" inside it (e. g. our Bohmian friends [14]) will hardly accept the new ontology. But we must remember the Occam's razor motivation. Another objection is that, once we have the computer, we still need to provide it with the software. True: but this is the same challenge of grand unification of quantum field theory, and here at least we have a simple common "programmable" background, and we may hope to find unification in new kinds of principles, related e. g. to the topological nature of the network. The principles must be simple: the software must be simple. But the computational grand-unification, being naturally a lattice theory, would also have the great bonus of avoiding all problems due to the continuum which plague quantum field theory (ultraviolet divergences, the Feynman path integral, non localizability of measurements, and many more). On the other hand, the digital theory will likely miss some of the simplicity of the continuum, whence finding easy ways to interpolate digital with analog must have top priority.

Recovering the whole Physics as emergent from the quantum information processing is a large program: we need to build up a complete dictionary that translates all physical notions into information-theoretic words. And we want more than that: we want to know if the digital character of Reality is experimentally detectable at some scale.

In the following, I will briefly explore how physical notions, including space-time, can emerge from the quantum computation, and how the quantum-digital nature of Reality leads to physical consequences that are in-principle detectable.

Emergence, emergence, emergence.

Let's investigate the idea that the current quantum field theory is indeed a kind of "thermodynamic" limit, valid at the Fermi scale, of a deeper theory that holds at the Planck scale, where the quantum field is replaced by a giant quantum computer. We'll see that the free-flow of quantum information is described by a digital version of the Dirac equation, and this also provides informational interpretations for inertial mass and Planck constant. At the same time, the notion of Hamiltonian is emerging, and, the field can be eliminated in favor of pure qubits. Some of these ideas for the moment plainly work in one space dimension, and are only a starting point: later in the paper we will see routes to be explored for larger space dimensions.

The free flow of information is the Dirac equation. One striking feature of the computational paradigm is that Lorentz covariance is a free bonus. As a matter of fact, Lorentz covariance must emerge from the computation if this is able to simulate Reality. And, the Dirac equation turns out to be just the free flow of quantum information [15].

As mentioned, we will restrict to one space dimension, and discuss larger dimensions later. In the quantum computer information can flow in a fixed direction only at the maximum speed of one-gate-per-step. In the digital world there is no physical unit: time and space are measured by counting, and the digital-analog conversion factors will be given by a time τ expressed in seconds and length a expressed in meters, which can be interpreted as the minimal space distance and time-distance between events, respectively. We may think to a as providing the Planck scale, namely 0.1 mm compared to an electron as huge as an entire galaxy! In analog units, the maximal speed is then given by $c = a/\tau$.

Mathematically we describe the information flow in the two directions by the two field operators ϕ^+ and ϕ^- for the right and the left propagation, respectively. In equations:

$$\widehat{\partial}_t \begin{bmatrix} \phi^+ \\ \phi^- \end{bmatrix} = \begin{bmatrix} c\widehat{\partial}_x & 0 \\ 0 & -c\widehat{\partial}_x \end{bmatrix} \begin{bmatrix} \phi^+ \\ \phi^- \end{bmatrix}, \tag{1}$$

where the hat on the partial derivative denotes that they are finite-difference generally extended to multiple events. If we take the maximal information speed c as a universal constant, then c must be the speed of light. Now, the only way of slowing-down the information flow is to have it changing direction repeatedly. A constant average speed corresponds to a simply periodic change of direction, which is described mathematically by a coupling between ϕ^+ and ϕ^- with an imaginary constant. Upon denoting by ω the angular frequency of such periodic change of direction, we have

$$\widehat{\partial}_t \begin{bmatrix} \phi^+ \\ \phi^- \end{bmatrix} = \begin{bmatrix} c\widehat{\partial}_x & -i\omega \\ -i\omega & -c\widehat{\partial}_x \end{bmatrix} \begin{bmatrix} \phi^+ \\ \phi^- \end{bmatrix}. \tag{2}$$

This is just the Dirac equation without spin. As we will see later, the coupling between the two fields will introduce a renormalization of c due to unitariety. The spin is recovered by noticing that the quantum computational network must have alternate rows of gates A and B [15], e. g. as in Fig. 1, whence one needs to merge diamond shaped sets of four connected events into a single coarse-grained event in order to keep the same network topology. The coarse-grained events are now connected by duplex wires, and such system-doubling corresponds to spin. As we will see later, the event-merging is also needed for more fundamental topological reasons in order to build-up digital coordinates, whence the spin may have a purely topological origin.

Inertial mass and Planck constant. The slowing down of information propagation due to the zig-zag can be considered as the emergent informational meaning of *inertial mass*, quantified by the angular frequency ω. The analogy with the Dirac equation leads us to write the coupling constant in terms of the Compton wavelength $\lambda = c\omega^{-1} = \hbar/(mc)$ (which is just the identity $m = c^{-2}\hbar\omega$ between the Planck quantum and the rest-energy), which can be regarded as a reinterpretation of the Planck constant $\hbar$ as the conversion factor between the informational notion of inertial mass in sec^{-1} and its customary notion in Kg.

Minkowski space-time. We have seen that Lorentz covariance is a free-bonus of the quantum computation. But how covariance emerges from the computation? The way to understand the mechanism is to define time in terms of a global clock for synchronizing the parallel distributed computation [16]. Giving a rule for establishing which subroutines are called at the same time according to the global clock corresponds to build up a *foliation* on the circuit,

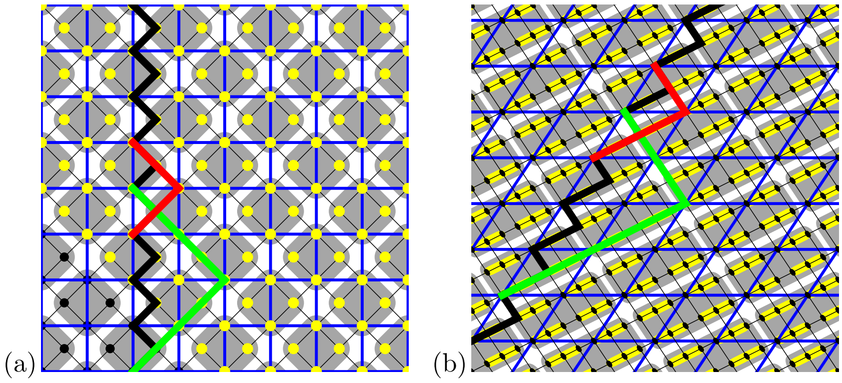

Figure 2. Digital coordinate systems built up by sending information back and forth between events (see text), in the rest frame (a), and in a boosted frame (b) (from Ref. [19]). The zig-zag line represents the synchronization clock. Since the time-precision is the duration of the entire tick-tack there are events that cannot be discriminated, whence they must be grouped into sets that share the same digital coordinates.

each leaf representing space at a different time. Uniform foliations corresponds to "boosts", namely inertial frames. The digital analog of the Lorentz space-contraction and time-dilation thus emerge in terms of an increased density of leaves and a decreased density of events per leaf in the boosted frame [17].

In order to build up a coordinate system digitally, information must be sent back and forth between subroutines at the maximum speed (light-signals). As shown in Fig. 2a, when the signal reaches back the clock (represented by the zig-zag) one can establish the time of the remote event as the intermediate time between the instant the signal has been sent and the instant it has been received. In this way one also finds out the distance of the event, thus building up the full coordinate system. Since the minimum time-interval is the duration of the clock tick-tack, there are events that cannot be discriminated, whence are grouped into sets sharing the same digital coordinates, in such a way that the new "thick events" are linked together within a network with the same topological structure of the original one. As shown in Ref. [18], in this way one builds up a "digital" version of the Lorentz transformations. In the rest frame this procedure leads to merging diamond shaped sets of four connected events into a single coarse-grained event (see Fig 2a). The coarse-grained events are now connected by duplex wires: as mentioned, this may be regarded as a topological motivation for spin [19].

In the digital-analog conversion, in addition to the continuous interpolation, one needs to rescale the digital coordinates by a constant factor χ [18] corresponding to the "thickness" of the event measured as the square-root of the number of diamonds inside the event ($\chi = \sqrt{3}$ in Fig. 2b). As we will see later, the coarse-graining of events plays a crucial role for $D > 1$ space-dimensions.

Hamiltonian. Differently from quantum field theory, in the quantum computer there is no Hamiltonian. All gates produce transformations far from the identity, otherwise we would need unbounded maximum speed for the information flow in order to get finite average information speed, recovering Einstein causality only in a continuum limit, as in the Lieb-Robinson bound [20]. The Hamiltonian, however, becomes an emergent notion for linear field evolution, as for the free flow of the Dirac equation (2). The Hamiltonian emerges as a difference between unitary matrices [15], and it has the form[1]

$$H = -\frac{i}{k\tau}\sum_n \phi_n^\dagger (U_f - U_b^\dagger)\phi_n, \tag{3}$$

where ϕ_n denotes the vector field (ϕ_n^+, ϕ_n^-) in Eq. (2) at the nth system, $2k$ is the number of steps to evaluate the time-difference derivative $\widehat{\partial}_t$ in Eq. (2), and $U_f(U_b\ )$ denotes the unitary matrix of the k-step forward(backward) evolution of the field. The homogeneity in time of the quantum-circuit guarantees that there exists a periodicity k of circuit rows such that $U_b = U_f$, whence the local Hamiltonian is an Hermitian matrix. Therefore, in the digital framework the Hermiticity of the field Hamiltonian is a consequence of the constancy of the physical law, i. e. the periodicity of the network, whereas looking at the tiniest scale ($k = 1$) Hermiticity may be lost.

Eliminating fields and using only qubits. In a quantum-digital world there are only qubits or quantum harmonic-oscillators unitarily interacting: everything—including not only the classical description, but also the quantum field itself—must be emergent. The quantum field must then be eliminated from the framework, being just a relic of the field quantization rules. In the Bose case the field is simply equivalent to a local harmonic-oscillator operator. For the Fermi case the field can be eliminated for $D = 1$ space dimensions using a simple algebraic construction in terms of local qubits [2].

Particles and quantization rules. For the "vacuum" state one can consider any state left invariant by the evolution, e. g. the state with all qubits in the state "0" (or all Bose-oscillators in the ground state). It is easy to see that for the Dirac equation the number N of qubits in the state "1" (or the total number of oscillator excitations in the Bose case) is a constant of motion. N can be interpreted as a number of particles. For nonzero mass a single localized particle would immediately spread up into a superposition of quantum zig-zags along the circuit. In a way analogous to the construction of the Hamiltonian one can define particle position and momentum (delocalized particles), and classical theory would emerge through "typical" trajectories of qubits in the state "1".

What about more than one space-dimensions? Many things that we have seen have been derived for a single space-dimension. For larger dimensions some assertions made are not valid: for example the assertion that information must flow at the maximum speed when the direction of flow is fixed, whereas it must zig-zag in order to slow-down. This is not true for e.g. in a BCC $(1+2)$-dimensional lattice (see Fig. 3), where in some directions the fastest path is a zig-zag, with a slowing down by a factor $\sqrt{M}$, M integer (but there are indeed M different fastest paths in quantum superposition!). Are there remedies to this problem?

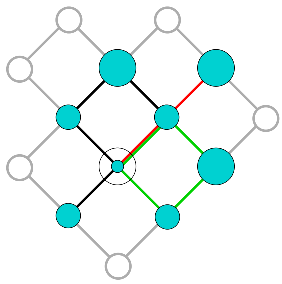

Figure 3. BCC $(1+2)$-dimensional computational network: top view. Information must zig-zag to flow at the maximal speed in diagonal direction (green lines). This leads to a slow-down of $\sqrt{2}$ factor of the analog speed compared to the red-line direction.

A first possible cure is changing the shape of the circuit, considering e. g. a network in form of a Regge triangulation with tetrahedra (gates at the vertices, and three inputs wires on the bottom and three output wires on the top). A top view of such network is shown in Fig. 4, where one can see how the maximal speed is the same in all directions, and is always achieved by a zig-zag motion.

A second possibility hinges on the crucial role played by the coarse-graining of events. For $D > 1$ the point-like nature of events makes their density change discontinuously with the reference system, as e.g. the site-density on a plane rotating around a principal crystal axis in a BCC lattice. But, as we have seen, the coarse-graining provides an intrinsic "thickness" for events, which smooths the discontinuous changes of their density. Roughly speaking, the thickness of events cures the anisotropy of the lattice in a way similar to what happens in crystals, e. g. a crystal with cubic symmetry macroscopically looks isotropic.

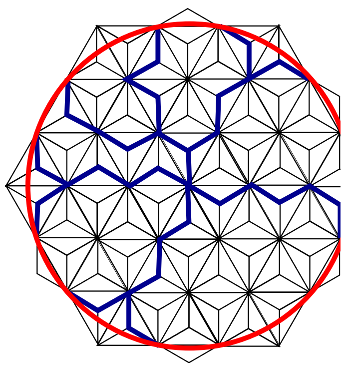

Figure 4. In a computational network made with tetrahedra (Regge triangulation) the maximal information speed is the same in all directions, and is always achieved by a zig-zag.

Another point where we found an apparent obstacle for generalization to $D > 1$ was the elimination of the field in favor of the local qubits, which for the Fermi field seems possible only for $D = 1$, whereas it is perfectly fine for all D for the Bose case. We should however remind that, differently from the Bose field, the Fermi field is not measurable in any sense (it has no coherent states, nor a classical counterpart, since in the path integral it is a Grassman field).

The apparent obstructions to digitalization of Reality for $D > 1$ are very fascinating. Will their solution open yet unknown features of the quantum fabric of space-time? Or will they disprove the existence of a quantum-digital reality?

Can we experimentally see the digital nature of Reality?

Clearly, the digital framework should be in principle detectable through violations of Lorentz covariance, e. g. at huge boosts, where one would have so extreme space-contractions that would go below the minimum distance between events. Or else the Planck length or energy scale must be invariant along with c, as in the doubly special relativity of Amelino-Camelia [21] or in the deformed Lorentz transformations of Magueijo and Smolin [22]. Another possibility is to see the truly "quantum-digital" nature of Reality. As shown in Ref. [15], the discreteness of the quantum circuit along with unitariety of quantum theory, will be responsible for a new phenomenon: the occurrence of a mass-dependent refraction index of the vacuum. Such refraction index comes from the requirement that the maximum information speed a/τ is bounded, in order to guarantee the emergence of special relativity from the computation, contrarily to the case e. g. of the Lieb-Robinson bound [20], where Einstein causality is recovered it in the continuum limit. For the digital Dirac Eq. (3) the refraction index is[3]

$$\frac{1}{n} = \sqrt{1 - \left(\frac{m}{M}\right)^2}, \tag{4}$$

where $M = \hbar/(2ac)$. If we take the distance $2a$ between neighboring gates equal to the Planck length (two qubits per Planck length), M is the Planck mass. According to Eq. (4) the information flow halts at the Planck mass: how this may be related to the holographic principle, is a mystery.

What about Gravity?

The big question is now where gravity comes from. At this early stage of this quantum-digitalization program I can only hypothesize possible lines of research. A very appealing possibility is to believe in a strong version of the equivalence principle, i. e. that inertial and gravitational masses are actually the same informational entity. This means that gravity must be a quantum effect! This idea is not new, and has been considered by Andrei Sakharov with his *induced gravity* of [23]. The work of Seth Lloyd [24] is also in this spirit. Literally the idea that gravity is a quantum effect means that it should be exhibited at the level of free field, whence such an effect should be truly a manifestation of the digital nature of the field, a low-order digital correction to the analog free-field theory (a possible mechanism is speculated in Ref. [25]). This way of looking at gravity is also deeply connected to the Erik Verlinde's idea of gravity as an entropic force [26], being the holographic principle the result of the "conservation" of quantum information—a founding axiom of Quantum Theory [10]. A second possibility—this one more in the old spirit of Einstein theory—is to consider a computational network with dynamical causal connections, e. g. "programmed" by a parallel circuit, as a digital gauge-field. This can be also regarded as a kind of third-quantization procedure, in which the causal links–the *systems*—become themselves quantum states of some higher-level kind of systems.

Notes

[1] Clearly, the matrix of the linear transformation of fields must be unitary, and this will also guarantee preservation of (anti)commutation relations for the field. Upon denoting by $\widehat{\partial}_t\phi := (2k\tau)^{-1}[\phi(+k\tau) - \phi(-k\tau)]$ the coarse-grained finite-difference over $2k$ time-steps, one can define a *local* Hamiltonian matrix as the one providing the (discrete) time-derivative locally of the field, namely $\mathrm{H}_{\mathrm{loc}}\phi_n := i\widehat{\partial}_t\phi_n$ (here ϕ_n denotes the vector (ϕ_n^+, ϕ_n^-) of the field operators at nth system), whence $\mathrm{H}_{\mathrm{loc}} = \frac{i}{k\tau}(U_f - U_b^\dagger)$, where $U_f (U_b\ )$ denotes the unitary matrix of the k-step forward(backward) evolution of the field. Then, it is immediate to check that the field operator $H = -\sum_n \phi_n^\dagger \mathrm{H}_{\mathrm{loc}}\phi_n$ works in all respects as a field Hamiltonian, since it satisfies the field dynamical equation $i\widehat{\partial}_t\phi_n = [H, \phi_n]$.

[2] Using the Jordan Schwinger realization of unitary matrices it is easy to show that the unitary operators of the gate can be written as exponential of bilinear forms of the field operators involved by the gate. The field operators in turn can be easily written as local operators in the Bose case, e. g. $\phi_n^+ = a_{2n}$ and $\phi_n^- = a_{2n+1}$, with a_l harmonic-oscillator operators $[a_l, a_k^\dagger] = \delta_{lk}$. In the Fermi case, however, the field is anticommuting, whence it is a nonlocal operator. For $D = 1$ space dimension, however, as shown in Ref. [15] one can use the Clifford algebraic construction $\phi_n^+ = \gamma_{2n}$, $\phi_n^- = \gamma_{2n+1}$, with $\gamma_l = \sigma_l^- \prod_{k=-\infty}^{l-1} \sigma_k^3$, and find that the unitary operators of gates are bilinear functions only of the local algebras of their wires. We thus have a quantum computer which really evolves just local qubits, and the fields are eliminated. An analogous construction for $D > 1$ is unknown.

[3] A lower bound for the refraction index holding for any circuit can be established as follows. As said, for sufficient coarse-graining in time the Hamiltonian is Hermitian, whence $U_f = U_b$. We thus have $\mathrm{H}_{\mathrm{loc}} = \frac{i}{2k\tau}(U_f - U_f^\dagger)$ (k the number of steps of the coarse-graining). The norm is bounded as $\|\mathrm{H}_{\mathrm{loc}}\| \leq (2\tau)^{-1}$. In addition to a free-propagation term in $\mathrm{H}_{\mathrm{loc}}$ of the form $i\frac{c}{n}\sigma^3\widehat{\partial}_x$, we take any Hermitian operator constant in x with the dimension of an inverse time. Upon denoting by cl^{-1} the maximum eigenvalue of such operator, the norm of $\mathrm{H}_{\mathrm{loc}}$ is thus obtained by Fourier transform at wave-vector $k = \pi/(4a)$, and the norm bound becomes $n^{-2} + 4a^2l^{-2} \leq 1$, namely $n^{-1} \leq \sqrt{1 - \left(\frac{2a}{l}\right)^2}$. For the Dirac equation (3) the bound is satisfied with the equality [15], and $l \equiv \lambda$ is the Compton wavelength.

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