Optimal phase-covariant cloning for qubits and qutrits

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We consider cloning transformations of equatorial qubits $|\psi_\phi\rangle = 1/\sqrt{2}(|0\rangle + e^{i\phi}|1\rangle)$ and qutrits $|\psi_{\phi,\theta}\rangle = 1/\sqrt{3}(|0\rangle + e^{i\phi}|1\rangle + e^{i\theta}|2\rangle)$, with the transformation covariant for rotation of the phases ϕ and θ. The optimal cloning maps are derived without simplifying assumptions from first principles, for any number of input and output qubits, and for a single-input qutrit and any number of output qutrits. We also compare the cloning maps for global and single-particle fidelities, and we show that the two criteria lead to different optimal maps.

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I. INTRODUCTION

The impossibility of perfectly cloning unknown quantum states selected from a nonorthogonal set is a typical quantum feature [1], and is the basis of the security of quantum cryptography [2–4]. In fact, the potential eavesdropper Eve cannot clone the quantum state transmitted by Alice, recover it from multiple copies, and retransmit it undisturbed to the receiver Bob. Eve, however, can try to realize an approximate cloning [5–8] in an optimal way, maximizing the fidelity of the copies with the original state, and this is a possible eavesdropping strategy. The eavesdropping strategies that are known to be optimal so far are actually based on cloning attacks [9–11]. Moreover, quantum cloning allows to study the sharing of quantum information among several parties and it may be applied also to study the security of multiparty cryptographic schemes [12].

Generally the values of the fidelity achieved by optimal cloning transformations depends on the set of allowed input states. In particular, higher fidelities can be achieved for smaller sets of input states, since the more information about the input is given, the better the input states can be cloned. More precisely, for group-covariant cloning [13]—where the set of input states is the orbit of a given state under the action of a group of unitary transformations—the smaller is the group the higher is the fidelity averaged over the input states.

In this paper we will develop a thorough analysis of the cloning map that is optimal for equatorial qubits and qutrits without any simplifying assumptions, including that of group covariance and the requirement that the output of the cloning map has support on a symmetric tensor-product Hilbert space. As we will see in Sec. II, these assumptions can be derived from the form of the fidelity that one wants to maximize. More precisely, we will derive with no assumption the optimal quantum cloning transformation maximizing the fidelity averaged uniformly over all states of a Bloch sphere equator

$$|\psi_\phi\rangle = \frac{1}{\sqrt{2}}[|0\rangle + e^{i\phi}|1\rangle], \tag{1}$$

where $\{|0\rangle,|1\rangle\}$ represent a basis for a qubit and the parameter $\phi \in [0,2\pi)$ is the angle between the Bloch vector and the x axis, with the equator in the x-y plane with the Bloch sphere. As we will see, such averaged form for the fidelity automatically leads to the optimal cloning covariant under the Abelian group $U(1)$ of phase rotations—the so-called *phase-covariant* cloning [9]. After the first analysis of Ref. [9], where only the upper bounds for the fidelity were derived by exploiting a connection between optimal phase-covariant cloning and phase estimation, in Refs. [13,14] a value for the fidelity that breached the bound given in Ref. [9] was found for the $1\to3$ cloning, apparently obtained under the same assumptions [15]. Then in Ref. [14] a cloning transformation from an arbitrary number of input copies N to an arbitrary number M of output copies was presented, however, it was proved to be optimal only for $N=1$. In this paper we will prove that the cloning maps of Ref. [14] are generally suboptimal for $N>1$ and we will derive the optimal ones for any values of N and M. In the derivation of the optimal cloning maps we will use the general method designed for group-covariant cloning introduced in Ref. [13], which exploits the correspondence between CP (completely positive) maps and positive operators. We also extended our analysis to the case of phase-covariant cloning for *qutrits*, namely, for quantum states with dimension $d=3$. Here the covariance group $U(1)\times U(1)$ is still Abelian, and describes the rotation of two different phases.

The paper is organized as follows. In Sec. II we describe the general theory of optimal phase-covariant cloning, giving the definitions of all relevant quantities in the qubit case, since for the qutrit case the treatment is strictly analogous. The starting point is the maximization of a phase-averaged fidelity, which will lead to a phase-covariant CP map with output on the symmetric Hilbert space of the output copies. Then the theory of group-covariant cloning of Ref. [13] is shortly reviewed and specialized to the case of phase

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covariance. In Sec. III we derive the optimal phase-covariant cloning for qubits for any number of input and output copies, giving the fidelities for all cases. In Sec. IV the same derivation is given for qutrits with any number of output copies, starting from a single input copy. Finally, in Sec. V we conclude with a discussion of the results, and with some open problems and future perspectives.

II. OPTIMAL PHASE-COVARIANT CLONING

A cloning map is a special kind of *quantum channel*, i.e. a trace-preserving CP map. In the cloning case, the CP map $\mathcal{C}$ goes from input states in $\mathcal{H}$ to output states in $\mathcal{H}^{\otimes M}$, with the output state invariant under the permutations of the M output spaces. More generally, if we have $N>1$ identical copies available, the map goes from an input state $\rho^{\otimes N}$ on the input Hilbert space $\mathcal{H}_{in}$ given by the symmetric subspace $\mathcal{N}\doteq(\mathcal{H}^{\otimes N})_+$ of the tensor product $\mathcal{H}^{\otimes N}$ to the output space $\mathcal{M}\doteq\mathcal{H}^{\otimes M}$ with $M>N$, and with the output state permutation invariant. Actually, as we will see in the following, the optimal map itself will have the output state restricted to the symmetric subspace $\mathcal{M}\equiv(\mathcal{H}^{\otimes M})_+$, even though, generally, permutation invariance of the state does not imply that the state has support in the symmetric subspace. In the following we will denote a cloning map from N to M "copies" as $\mathcal{C}_{NM}$.

We want to find a cloning map $\mathcal{C}_{NM}$ that minimizes the following averaged fidelity:

$$\overline{f}[\mathcal{C}_{NM}]=\int_0^{2\pi}\frac{\mathrm{d}\phi}{2\pi}f[\mathcal{C}_{NM}](\phi),$$

$$f[\mathcal{C}_{NM}](\phi)=\mathrm{Tr}[|\psi_\phi\rangle\langle\psi_\phi|^{\otimes M}\mathcal{C}_{NM}(|\psi_\phi\rangle\langle\psi_\phi|^{\otimes N})], \quad (2)$$

for *equatorial qubit* input states $|\psi_\phi\rangle$ defined as

$$|\psi_\phi\rangle=\frac{1}{\sqrt{2}}(|0\rangle+e^{i\phi}|1\rangle). \quad (3)$$

We will call the cloning map *phase covariant* if it satisfies the following covariance relation:

$$\mathcal{C}_{NM}(U_\phi^{\otimes N}\rho_{\mathcal{N}}U_\phi^{\dagger\otimes N})=U_\phi^{\otimes M}\mathcal{C}_{NM}(\rho_{\mathcal{N}})U_\phi^{\dagger\otimes M}, \quad (4)$$

where U_ϕ is the unitary phase rotation operator

$$U_\phi=\exp\left[\frac{i}{2}\phi(1-\sigma_z)\right], \quad (5)$$

$\sigma_{x,y,z}$ denotes the usual Pauli matrices, and $\rho_{\mathcal{N}}$ is any state in $\mathcal{N}\equiv(\mathcal{H}^{\otimes N})_+$. In particular, according to the fidelity in Eq. (2), we will consider only input states of the form

$$\rho_{\mathcal{N}}=|\psi_0\rangle\langle\psi_0|^{\otimes N}. \quad (6)$$

The unitary transformation in Eq. (5) gives the phase shift $U_\phi|\psi_{\phi'}\rangle=|\psi_{\phi'+\phi}\rangle$. For qutrits the situation will be analogous, with input states of the form

$$\rho_{\mathcal{N}}=|\psi_{0,0}\rangle\langle\psi_{0,0}|^{\otimes N}, \quad (7)$$

where

$$|\psi_{\phi,\theta}\rangle=\frac{1}{\sqrt{3}}(|0\rangle+e^{i\phi}|1\rangle+e^{i\theta}|2\rangle) \quad (8)$$

denotes an equatorial qutrit state, and in place of U_ϕ we will consider the two-phase rotation operator $U_{\phi,\theta}$ that achieves the phase shift $U_{\phi,\theta}|\psi_{\phi',\theta'}\rangle=|\psi_{\phi'+\phi,\theta'+\theta}\rangle$.

Upon defining the rotated map $\mathcal{C}_{NM}^{\phi}$ as follows:

$$\mathcal{C}_{NM}^{\phi}(\rho_{\mathcal{N}})\doteq U_\phi^{\dagger\otimes M}\mathcal{C}_{NM}(U_\phi^{\otimes N}\rho_{\mathcal{N}}U_\phi^{\dagger\otimes N})U_\phi^{\otimes M}, \quad (9)$$

from Eq. (4) we see that covariance of the map $\mathcal{C}_{NM}$ is equivalent to the identity $\mathcal{C}_{NM}\equiv\mathcal{C}_{NM}^{\phi}$ for every ϕ. Since the fidelity $f[\mathcal{C}_{NM}](\phi)$ is linear versus the cloning map $\mathcal{C}_{NM}$, the averaged fidelity in Eq. (2) can be also written in the form

$$\overline{f}[\mathcal{C}_{NM}]=\int_0^{2\pi}\frac{d\phi}{2\pi}f[\mathcal{C}_{NM}^{\phi}](0)\equiv f[\overline{\mathcal{C}_{NM}^{\phi}}](0), \quad (10)$$

where clearly $f[\mathcal{C}_{NM}^{\phi}](0)\equiv f[\mathcal{C}_{NM}](\phi)$, and the averaged map $\overline{\mathcal{C}_{NM}^{\phi}}$ is obviously defined as

$$\overline{\mathcal{C}_{NM}^{\phi}}=\int_0^{2\pi}\frac{d\phi}{2\pi}\mathcal{C}_{NM}^{\phi}. \quad (11)$$

Since, by definition, the averaged map $\overline{\mathcal{C}_{NM}^{\phi}}$ is phase covariant, Eq. (10) simply means that the cloning map minimizing the averaged fidelity (2) must itself be covariant. Therefore, finding the optimal cloning map $\mathcal{C}_{NM}$ that minimizes the fidelity (2) is equivalent to find the optimal phase-covariant map $\mathcal{C}_{NM}$ that minimizes the following fidelity:

$$f_{NM}\doteq f[\mathcal{C}_{NM}](0)=\mathrm{Tr}[|\psi_0\rangle\langle\psi_0|^{\otimes M}\mathcal{C}_{NM}(|\psi_0\rangle\langle\psi_0|^{\otimes N})]. \quad (12)$$

Moreover, due to orthogonality with the state $|\psi_0\rangle\langle\psi_0|^{\otimes M}$, any component of the output state $\mathcal{C}_{NM}(|\psi_0\rangle\langle\psi_0|^{\otimes N})$ which is not supported on the symmetric subspace $(\mathcal{H}^{\otimes M})_+$ will give no contribution to the fidelity (12). Therefore, there will be always an optimal cloning map having output on the symmetric space $(\mathcal{H}^{\otimes M})_+$, and in the following we can restrict our attention to such maps only, and take $\mathcal{M}\equiv(\mathcal{H}^{\otimes M})_+$. We will also consider for comparison the average single particle fidelity

$$F_{NM}=\frac{1}{M}\mathrm{Tr}[(|\psi_0\rangle\langle\psi_0|\mathbb{1}^{\otimes M-1}+\mathbb{1}|\psi_0\rangle\langle\psi_0|\mathbb{1}^{\otimes M-2}+\cdots+\mathbb{1}^{\otimes M-1}|\psi_0\rangle\langle\psi_0|)|\psi_0\rangle\langle\psi_0|^{\otimes N}\mathcal{C}_{NM}(|\psi_0\rangle\langle\psi_0|^{\otimes N})]. \quad (13)$$

As shown in Ref. [13], it is convenient to study covariant CP maps in terms of invariant positive operators which are in one-to-one correspondence with CP maps. In the present context this means to consider the positive operators R_{NM} defined as

$$R_{NM}=\mathcal{C}_{NM}\otimes\mathcal{I}_{\mathcal{N}}(|I\rangle\rangle\langle\langle I|), \quad (14)$$

where $\mathcal{I}_{\mathcal{N}}$ denotes the identity map over $\mathcal{N}\equiv(\mathcal{H}^{\otimes N})_+$, and $|I\rangle\rangle$ is the maximally entangled vector on $\mathcal{N}\otimes\mathcal{N}$,

$$|I\rangle\rangle=\sum_{n=0}^{N}|s_{N,n}\rangle\otimes|s_{N,n}\rangle, \tag{15}$$

$\{|s_{N,n}\rangle\}$ denoting any orthonormal basis for $(\mathcal{H}^{\otimes N})_+$, that we conveniently choose as follows:

$$|s_{N,n}\rangle=C(N,n)^{-1/2}\sum_j P_j^{(N)}|\underbrace{00\ldots0}_{N-n}\,\underbrace{11\ldots1}_{n}\rangle, \tag{16}$$

where $\{P_j^{(N)}\}$ denote the permutation operators of the N qubits, and $C(N,n)$ is the binomial coefficient $C(N,n)\equiv N!/n!(N-n)!$. As shown in Ref. [13], one can see that R_{NM} is a positive operator on $\mathcal{M}\otimes\mathcal{N}$, which is in one-to-one correspondence with the CP-map $\mathcal{C}_{NM}$, with the trace-preserving condition for the map writing in terms of the operator R_{NM} as follows:

$$\mathrm{Tr}_{\mathcal{M}}[R_{NM}]=I_{\mathcal{N}}. \tag{17}$$

The map $\mathcal{C}_{NM}$ can be recovered from the positive operator R_{NM} as follows:

$$\mathcal{C}_{NM}(\rho_{\mathcal{N}})=\mathrm{Tr}_{\mathcal{N}}[(I_{\mathcal{M}}\otimes\rho_{\mathcal{N}}^{\mathrm{t}})R_{NM}], \tag{18}$$

where O^{t} denotes the transposed operator of O with respect to the same orthonormal basis (15) chosen for the maximally entangled vector in Eq. (16), namely, one defines $O^{\mathrm{t}}\doteq(O^{\dagger})^*$ where the *complex conjugated* O^* of the operator O is defined as the operator having complex-conjugated matrix of the operator O with respect to the same orthonormal basis (15). Notice that for the particular state in Eq. (6), one has $\rho_{\mathcal{N}}^{\mathrm{t}}\equiv\rho_{\mathcal{N}}$, since $|\psi_0\rangle^{\otimes N}$ has all real coefficients on the basis (15). The covariance (4) of the CP map $\mathcal{C}_{NM}$ in terms of the operator R_{NM} becomes the invariance relation

$$[R_{NM},U_\phi^{\otimes M}\otimes(U_\phi^{\otimes N})^*]=0, \tag{19}$$

and in our case we have simply $(U_\phi^{\otimes N})^*\equiv U_{-\phi}^{\otimes N}$. Then, according to the Schur lemmas, the positive operator R_{NM} is given by the following direct sum:

$$R_{NM}=\oplus_\nu R_\nu, \tag{20}$$

where ν runs over all inequivalent unitary irreducible representations (UIR) contained in the reducible one $U_\phi^{\otimes M}\otimes(U_\phi^{\otimes N})^*$, with all equivalent representations grouped together, and with R_ν denoting any positive operator over the space of all representations equivalent to ν, with the overall constraint of the trace-preserving condition (17).

Our purpose is to find the optimal phase-covariant cloning map that maximizes the fidelity in Eq. (12), which using Eq. (18) can be rewritten in terms of the positive operator R_{NM} as follows:

$$f_{NM}=\mathrm{Tr}[(|\psi_0\rangle\langle\psi_0|^{\otimes M}\otimes|\psi_0\rangle\langle\psi_0|^{\otimes N})R_{NM}]. \tag{21}$$

The derivation for the case of qutrits will be strictly analogous to that of qubits.

III. OPTIMAL CLONING FOR QUBITS

Since the phase rotation group is abelian, all UIRs of the group are unidimensional. The inequivalent representations can be conveniently labeled by the non-negative integer ν, corresponding to the invariant spaces of vectors where the group action is equivalent to multiplication by the phase factor $\exp(i\nu\phi)$. Therefore, in the reduction of the representation $U_\phi^{\otimes M}\otimes U_{-\phi}^{\otimes N}$, each UIR equivalent to the representation ν is spanned by a vector of the type

$$|M-j-\nu,j+\nu\rangle\otimes|N-j,j\rangle,$$

$$j=0,\ldots,\min(N,M-\nu),$$

$$\nu=0,\ldots,M-N, \tag{22}$$

where $|N-j,j\rangle$ denotes a state of N qubits, where $N-j$ of them are in state $|0\rangle$, while the remaining j are in state $|1\rangle$.

We will now look for the optimal transformations, namely, the transformations that maximize the fidelity f_{NM}. As proved above, we can restrict our attention to the symmetric subspace, and therefore we will consider the equivalent representations corresponding to the symmetric states $\{|s_{M,j+\nu}\rangle|s_{N,j}\rangle,j=0,\ldots\min(N,M-\nu)\}_\nu$, where ν labels the inequivalent representations ($\nu=0,\ M-N$). In the evaluation of the fidelity we take $|\psi_0\rangle=(|0\rangle+|1\rangle)/\sqrt{2}$. The fidelity of the map is made of contributions of the form

$$\mathrm{Tr}[(|\psi_0\rangle\langle\psi_0|^{\otimes M+N})(|s_{M,j+\nu}\rangle\langle s_{M,k+\nu}|\otimes|s_{N,j}\rangle\langle s_{N,k}|)]$$
$$=\frac{1}{2^{N+M}}\sqrt{C(N,j)C(N,k)}\sqrt{C(M,j+\nu)C(M,k+\nu)}. \tag{23}$$

Each block of equivalent representations labeled by ν is given by the positive operator

$$R_\nu=\sum_{jk}r_{jk}^\nu|s_{M,j+\nu}\rangle\langle s_{M,k+\nu}|\otimes|s_{N,j}\rangle\langle s_{N,k}|, \tag{24}$$

where the trace-preserving condition for the operator R_{NM} leads to

$$\sum_{\nu=0}^{M-1}r_{jj}^\nu=1,\quad i=0,\ldots N. \tag{25}$$

Since each single contribution to fidelity (23) is positive versus j and k, the operators R_ν that maximize the fidelity have positive elements r_{jk}^ν and the off-diagonal terms are as large as possible, i.e., $r_{jk}^\nu=\sqrt{r_{jj}^\nu}\sqrt{r_{kk}^\nu}$. Therefore, the operator R_ν can be written as a (generally non normalized) projector $R_\nu=|r_\nu\rangle\langle r_\nu|$, where $|r_\nu\rangle=\Sigma_j r_j^\nu|s_{M,j+\nu}\rangle\otimes|s_{N,j}\rangle$, and $r_j^\nu\doteq\sqrt{r_{jj}^\nu}$.

Let us now explicitly construct the cloning map that optimizes the fidelity. We will first consider the simple case $N=1$. Each term R_ν will therefore give the following contribution to f_{1M}:

$$f_{1M}^\nu = \mathrm{Tr}[(|\psi_0\rangle\langle\psi_0|^{\otimes M}\otimes|\psi_0\rangle\langle\psi_0|)R_\nu]$$
$$= \frac{1}{2^{M+1}}(r_0^\nu\sqrt{C(M,\nu)}+r_1^\nu\sqrt{C(M,1+\nu)})^2 \qquad (26)$$

with $f_{1M}=\Sigma_\nu f_{1M}^\nu$.

For odd values of M the largest contribution to the fidelity comes from the projector with $\bar{\nu}=(M-1)/2$, because in this case both terms $\sqrt{C(M,\nu)}$ and $\sqrt{C(M,1+\nu)}$ are equal and are maximized simultaneously. Moreover, this contribution is maximized when the values of r_0 and r_1 are maximized, i.e. for $r_0^{(M-1)/2}=r_1^{(M-1)/2}=1$. In this case the optimal map is given by

$$R_{1M}=|r_{(M-1)/2}\rangle\langle r_{(M-1)/2}| \qquad (27)$$

and the fidelity takes the form

$$f_{1M}=\frac{1}{2^{M-1}}C(M,(M-1)/2). \qquad (28)$$

For even values of M the optimization procedure is more involved, because the coefficients $\sqrt{C(M,\nu)}$ and $\sqrt{C(M,1+\nu)}$ are different and cannot be maximized simultaneously by a single value of ν. In order to derive the form of the optimal map let us first notice that the same contribution f_{1M}^ν in Eq. (26) is also achieved by choosing $\nu'=M-\nu-1$ with $r_0^\nu=r_1^{M-\nu-1}$ and

$$r_1^\nu=r_0^{M-\nu-1}. \qquad (29)$$

Therefore we can look at contributions due to maps of the form

$$R'_\nu=\frac{1}{2}(R_\nu+R_{M-\nu-1}). \qquad (30)$$

By taking into account relations (29), in this case completeness constraint (17) can be written as

$$\sum_\nu (r_0^\nu)^2+\sum_\nu (r_1^\nu)^2=2. \qquad (31)$$

The optimal map is given by the values of ν that give the maximum contributions to Eq. (26), namely, for $\nu_-=M/2-1$ and $\nu_+=M/2$. Therefore the optimization problem consists in maximizing the quantity $r_0^{\nu_-}A+r_1^{\nu_-}B$, with the constraint $(r_0^{\nu_-})^2+(r_1^{\nu_-})^2=2$ and with $A=\sqrt{C(M,\nu_-)}$ and $B=\sqrt{C(M,\nu_-+1)}$. The solution is given by

$$r_0^{\nu_-}=\sqrt{2}\frac{A}{\sqrt{A^2+B^2}},\quad r_1^{\nu_-}=\sqrt{2}\frac{B}{\sqrt{A^2+B^2}}. \qquad (32)$$

Therefore, the optimal map R_{1M} for even values of M is given by

$$R_{1M}=\frac{1}{2}(|r_{\nu_-}\rangle\langle r_{\nu_-}|+|r_{\nu_+}\rangle\langle r_{\nu_+}|) \qquad (33)$$

with $r_0^{\nu_-}$ and $r_1^{\nu_+}$ given by Eq. (32), and $r_0^{\nu_+}=r_1^{\nu_-}$, $r_1^{\nu_+}=r_0^{\nu_-}$. The fidelity takes the form

$$f_{1M}=\frac{1}{2^M}C(M+1,M/2). \qquad (34)$$

Consider now the general case $N\to M$. Each contribution f_{NM}^ν to the fidelity takes the form

$$f_{NM}^\nu=\frac{1}{2^{N+M}}\left[\sum_{j=0}^{\min(N,M-\nu)} r_j^\nu\sqrt{C(N,j)C(M,j+\nu)}\right]^2. \qquad (35)$$

The maximum value is achieved for the representation $\bar{\nu}$ for which both the terms $\sqrt{C(N,j)}$ and $\sqrt{C(M,j+\nu)}$ are maximized at the same time. In fact, $\Sigma_\nu f_{NM}^\nu$ is a convex function of r_j^ν defined on convex domain (25), and the maximum is achieved on the extremal points $r_{jj}^\nu=1$ for some ν. This also corresponds to maximize the rhs. of Eq. (35) by adding "coherently" all the terms in the sum over j for a single value of ν. We have then to distinguish different cases: for N odd and M odd the simultaneous maximization of $\sqrt{C(N,j)}$ and $\sqrt{C(M,j+\nu)}$ occurs when $j=(N-1)/2$ and $\bar{\nu}=(M-N)/2$. In this case the optimal cloning map corresponds to $r_j^\nu=1$, and is described by

$$R_{NM}=|r_{(M-N)/2}\rangle\langle r_{(M-N)/2}|. \qquad (36)$$

The fidelity takes the explicit form

$$f_{NM}=\frac{1}{2^{N+M}}\left[\sum_{j=0}^{N}\sqrt{C(N,j)C(M,(M-N)/2+j)}\right]^2. \qquad (37)$$

An analogous argument and the results given in Eqs. (36) and (37) hold also when M and N are both even.

Consider now the case of even M and odd N, or vice versa. The two terms $\sqrt{C(N,j)}$ and $\sqrt{C(M,j+\nu)}$ are maximized at the same time for the two values $\nu_\pm=(M-N\pm1)/2$. In order to derive the optimal map we follow an argument analogous to the case $N=1$ discussed above. Actually, let us first notice that the same contribution f_{NM}^ν in Eq.

(35) is also achieved by choosing $\nu' = M-N-\nu$ with $r_j^{\nu} = r_{N-j}^{M-N-\nu}$. Therefore, as in the case $N=1$ we can look at cloning maps of the form

$$R'_{\nu} = \frac{1}{2}(R_{\nu} + R_{M-N-\nu}). \tag{38}$$

By exploiting the relation $r_j^{\nu} = r_{N-j}^{M-N-\nu}$, we can write the completeness condition as

$$\sum_{\nu} (r_j^{\nu})^2 + \sum_{\nu} (r_{N-j}^{\nu})^2 = 2, \quad j = 0, \ldots N/2. \tag{39}$$

As mentioned above, the greatest contributions to ther fidelity are given by the blocks with $\nu_{\pm} = (M-N\pm 1)/2$. The optimal cloning map will therefore be of the form

$$R_{NM} = \frac{1}{2}(R_{\nu_-} + R_{\nu_+}), \tag{40}$$

with the constraints $r_j^{\nu_+} = r_{N-j}^{M-N-\nu_-}$ and $(r_j^{\nu_-})^2 + (r_{N-j}^{\nu_-})^2 = 2, j = 0,..N/2$. The optimization of fidelity (35) with constraints (39) leads to the following solutions:

$$r_j^{\nu_-} = \sqrt{2}\frac{\sqrt{C(M,j+\nu_-)}}{\sqrt{C(M,j+\nu_-) + C(M,N-j+\nu_-)}},$$

$$r_{N-j}^{\nu_-} = \sqrt{2}\frac{\sqrt{C(M,N-j+\nu_-)}}{\sqrt{C(M,j+\nu_-) + C(M,N-j+\nu_-)}}. \tag{41}$$

Let us now consider as a quality criterion to optimize the cloning map the optimization of the average single-particle fidelity F_{NM}, defined as

$$F_{NM} = \frac{1}{M}\mathrm{Tr}[(|\psi_0\rangle\langle\psi_0|\mathbb{1}^{\otimes M-1} + \mathbb{1}|\psi_0\rangle\langle\psi_0|\mathbb{1}^{\otimes M-2} + \cdots + \mathbb{1}^{\otimes M-1}|\psi_0\rangle\langle\psi_0|) \otimes |\psi_0\rangle\langle\psi_0|^{\otimes N} R_{NM}]. \tag{42}$$

In this case we assume that the operators R_{NM} are supported on the symmetric subspace $(\mathcal{H}^{\otimes M})_+$. Notice that the last requirement is now an assumption, because the argument after Eq. (12), valid for the global fidelity f_{NM}, does not hold for the average single-particle fidelity. In this case the fidelity of the map is made of contributions of the form

$$\begin{aligned}
&\mathrm{Tr}[(|\psi_0\rangle\langle\psi_0|\mathbb{1}^{\otimes M-1}|\psi_0\rangle\langle\psi_0|^{\otimes N})(|s_{M-j-\nu}\rangle\langle s_{M-k-\nu}| \otimes |s_{N-j}\rangle\langle s_{N-k}|)] \\
&= \frac{1}{2^{N+1}}\left[C(N,j)\,\delta_{j,k} + \frac{\sqrt{C(N,j)C(N,j+1)}C(M-1,j+\nu)}{\sqrt{C(M,j+\nu)C(M,j+\nu+1)}}\,\delta_{j+1,k}\right] \\
&= \frac{1}{2^{N+1}}\left[C(N,j)\,\delta_{j,k} + \frac{1}{M}\sqrt{C(N,j)C(N,j+1)}\sqrt{(M-j-\nu)(j+\nu+1)}\,\delta_{j+1,k}\right],
\end{aligned} \tag{43}$$

where we have considered $k \geqslant j$. As in the case of the global fidelity f_{NM}, let us start from the case $N=1$.

Each term R_{ν} will therefore give the following contribution to F_{1M}:

$$F_{1M}^{\nu} = \frac{1}{4}\left[(r_0^{\nu})^2 + (r_1^{\nu})^2 + \frac{2}{M} r_0^{\nu} r_1^{\nu}\sqrt{(M-\nu)(\nu+1)}\right]. \tag{44}$$

For odd values of M the term $\sqrt{(M-\nu)(\nu+1)}$ is maximized for $\nu = (M-1)/2$. The optimal map, as in the case of the optimization of the global fidelity, is given by Eq. (27) with $r_0^{(M-1)/2} = r_1^{(M-1)/2} = 1$. The fidelity in this case takes the explicit form

$$F_{1M} = \frac{1}{2}\left(1 + \frac{M+1}{2M}\right). \tag{45}$$

For odd values of M, we can argue similarly to the case of the global fidelity, and therefore we have to maximize quantity (44) with the constraint $(r_0^{\nu_-})^2 + (r_1^{\nu_-})^2 = 2$. In this case the optimal solution corresponds to $r_0^{\nu_-} = r_1^{\nu_-} = 1$. The form of the optimal map is given by

$$R_{1M} = \lambda |r_{\nu_-}\rangle\langle r_{\nu_-}| + (1-\lambda)|r_{\nu_+}\rangle\langle r_{\nu_+}|, \tag{46}$$

with $0 \leqslant \lambda \leqslant 1$, and the fidelity takes the form

$$F_{1M} = \frac{1}{2}\left(1 + \frac{\sqrt{M(M+2)}}{2M}\right). \tag{47}$$

The above optimal single-particle fidelities are the same as those reported in Ref. [14], where cloning transformations restricted to the symmetric subspace were studied and the optimality of the single-particle fidelity was proved only for $N=1$.

Consider now the general case $N \to M$. Each contribution F_{NM}^{ν} to the fidelity takes the form

$$F_{NM}^{\nu}=\frac{1}{2^{N+1}}\Bigg[\sum_{j=0}^{\min(N,M-\nu)}(r_j^{\nu})^2 C(N,j)+\frac{2}{M}\sum_{j=0}^{\min(N,M-\nu)-1} r_j^{\nu} r_{j+1}^{\nu}\sqrt{C(N,j)C(N,j+1)}\times\sqrt{(M-j-\nu)(j+\nu+1)}\Bigg]. \tag{48}$$

The maximum value is achieved for the representation ν, for which both the terms $\sqrt{C(N,j)C(N,j+1)}$ and $\sqrt{(M-j-\nu)(j+\nu+1)}$ are maximized at the same time. We have then to distinguish different cases: for N odd and M odd this occurs when $j=(N-1)/2$ and $\nu=(M-N)/2$. In this case the optimal cloning map corresponds to $r_j^{\nu}=1$, and is described by

$$R_{NM}=|r_{(M-N)/2}\rangle\langle r_{(M-N)/2}|. \tag{49}$$

The fidelity in this case takes the explicit form

$$F_{NM}=\frac{1}{2}+\frac{1}{M2^N}\sum_{j=0}^{N-1}\sqrt{C(N,j)C(N,j+1)}\times\sqrt{[(M+N)/2-j][(M-N)/2+j+1]}. \tag{50}$$

The results given in Eqs. (49) and (50) hold also when M and N are both even. Notice that these results are in agreement with those conjectured in Ref. [14] for generic N and M.

Consider now the case where N and M have different parity, for example M is even and N is odd. The two terms $\sqrt{C(N,j)C(N,j+1)}$ and $\sqrt{(M-j-\nu)(j+\nu+1)}$ are maximized at the same time for the two values $\nu_{\pm}=(M-N\pm1)/2$. The optimal cloning map will therefore be of the form

$$R_{NM}=\frac{1}{2}(R_{\nu_-}+R_{\nu_+}), \tag{51}$$

with the constraints $r_j^{\nu_+}=r_{N-j}^{\nu_-}$ and $(r_j^{\nu_-})^2+(r_{N-j}^{\nu_-})^2=2, j=0,..N/2-1$. Therefore, we have to optimize fidelity (48) with $\nu=\nu_-$ by taking into account the above constraints, namely, the quantity

$$F_{NM}^{\nu_-}=\frac{1}{2}+\frac{1}{2^N M}\sum_{j=0}^{N-1} r_j^{\nu_-} r_{j+1}^{\nu_-}\sqrt{C(N,j)C(N,j+1)}\times\sqrt{(M-j-\nu_-)(j+\nu_-+1)}. \tag{52}$$

The forms of the coefficients r_j cannot be found in general. As an example we explicitly optimize the fidelity for $N=2$ and odd M. In this case $\nu_-=(M-3)/2$ and the form of the coefficients is given by

$$r_0^{\nu_-}=\sqrt{2}\frac{\sqrt{(M-1)(M+3)}}{\sqrt{(M-1)(M+3)+(M+1)^2}},$$

$$r_1^{\nu_-}=1,\quad r_2^{\nu_-}=\sqrt{2-(r_0^{\nu_-})^2}, \tag{53}$$

and the fidelity takes the explicit form

$$F_{2M}=\frac{1}{2}\left(1+\frac{\sqrt{M^2+2M-1}}{\sqrt{2}M}\right). \tag{54}$$

We can see that, in general when N and M have different parity the optimal solutions are not in agreement with the optimal transformations conjectured in Ref. [14].

We want to point out that the fidelity F_{NM} of the above optimal cloning transformations in the limit $M\to\infty$ coincides with the fidelity of optimal state estimation for N equatorial qubits [16].

Moreover, we want to stress that the cloning transformations that optimize the global fidelity coincide with the optimal ones for the single-particle fidelity only in the cases where N and M have the same parity.

IV. OPTIMAL CLONING FOR QUTRITS

In this section we will derive the optimal $1\to M$ cloning transformations for equatorial qutrit states

$$|\psi_{\phi,\theta}\rangle=\frac{1}{\sqrt{3}}(|0\rangle+e^{i\phi}|1\rangle+e^{i\theta}|2\rangle), \tag{55}$$

covariant under the group of rotations of both phases ϕ and θ. Again, since the group is abelian, all UIR of the group are unidimensional, and in a way analogous to the case of cloning of qubits, when restricting to output states supported on the symmetric subspace $(\mathcal{H}^{\otimes N})_+$, the equivalent UIR's are spanned by the vectors

$$|s_{M,\nu_1,\nu_2}\rangle|0\rangle,\quad |s_{M,\nu_1+1,\nu_2}\rangle|1\rangle,\quad |s_{M,\nu_1,\nu_2+1}\rangle|2\rangle, \tag{56}$$

where $\nu_1=0,\ldots,M-1$ and $\nu_2=0,\ldots,M-\nu_1-1$ label the invariant spaces of the UIR corresponding to multiplication by the phase factor $\exp(i\nu_1\phi+i\nu_2\theta)$, and $|s_{k,p,q}\rangle$ denotes the normalized symmetric state of k qutrits with $k-p-q$ qutrits in state $|0\rangle$, p in state $|1\rangle$ and q in state $|2\rangle$ (the state $|s_{k,p,q}\rangle$ is a superposition of $k!/(k-q-p)!p!q!$ orthogonal states).

In this case we will have contributions of the following type to the fidelity:

$$\mathrm{Tr}[|\psi_{0,0}\rangle\langle\psi_{0,0}|^{\otimes M+1}(|s_{M,\nu_1,\nu_2}\rangle\langle s_{M,\nu_1+1,\nu_2}|\otimes|0\rangle\langle1|)]=\frac{1}{3^{M+1}}\sqrt{T(M,\nu_1,\nu_2)}\sqrt{T(M,\nu_1+1,\nu_2)}, \tag{57}$$

where we define

$$T(M,\nu_1,\nu_2)=\frac{M!}{(M-\nu_1-\nu_2)!\,\nu_1!\,\nu_2!}. \tag{58}$$

Since all the above contributions are positive, we can apply the same argument as in the case of qubits and consider positive operators of the form $R_{\nu_1,\nu_2}=|r_{\nu_1,\nu_2}\rangle\langle r_{\nu_1,\nu_2}|$, where

$$|r_{\nu_1,\nu_2}\rangle = r_0^{\nu_1,\nu_2}|s_{M,\nu_1,\nu_2}\rangle|0\rangle + r_1^{\nu_1,\nu_2}|s_{M,\nu_1+1,\nu_2}\rangle|1\rangle + r_2^{\nu_1,\nu_2}|s_{M,\nu_1,\nu_2+1}\rangle|2\rangle, \tag{59}$$

and the trace-preserving condition for the operator R_{1M} leads to

$$\sum_{\nu_1,\nu_2}(r_0^{\nu_1,\nu_2})^2=\sum_{\nu_1,\nu_2}(r_1^{\nu_1,\nu_2})^2=\sum_{\nu_1,\nu_2}(r_2^{\nu_1,\nu_2})^2=1, \tag{60}$$

where in each sum ν_1 and ν_2 are constrained to give nonnegative entries in the states in Eq. (59).

Each operator R_{ν_1,ν_2} gives the following contribution to the fidelity

$$f_{\nu_1,\nu_2}=\frac{1}{3^{M+1}}[r_0^{\nu_1,\nu_2}\sqrt{T(M,\nu_1,\nu_2)} + r_1^{\nu_1,\nu_2}\sqrt{T(M,\nu_1+1,\nu_2)} + r_2^{\nu_1,\nu_2}\sqrt{T(M,\nu_1,\nu_2+1)}]^2. \tag{61}$$

The operator R_{ν_1,ν_2} that gives the highest contribution to the fidelity is that where the values of $T(M,\nu_1,\nu_2)$, $T(M,\nu_1+1,\nu_2)$, and $T(M,\nu_1,\nu_2+1)$ are maximized. This is easy to establish in the case of $M=3k+1$, because the three above expressions for T are all simultaneously maximized for $\nu_1=\nu_2=k$. Therefore, the optimal cloning map is given by $R_{k,k}$ with $r_0=r_1=r_2=1$. The corresponding fidelity takes the explicit form

$$f_M=\frac{1}{3^{M-1}}T\left(M,\frac{M-1}{3},\frac{M-1}{3}\right). \tag{62}$$

The cases with $M=3k$ and $M=3k+2$ are more involved, because the three values of T that appear in Eq. (61) cannot be maximized simultaneously. In order to find the form of the optimal maps we will follow an argument similar to the case of qubits. Notice first that the value of the contribution f_{ν_1,ν_2} to the fidelity does not change by performing any permutation of the basis states $\{|0\rangle,|1\rangle,|2\rangle\}$ for each of the $M+1$ qutrits in the operator R_{ν_1,ν_2}. This means that the three blocks labeled by (ν_1,ν_2), $(\nu_2,M-\nu_1-\nu_2-1)$, and $(M-\nu_1-\nu_2-1,\nu_1)$ give the same contribution to the fidelity. Therefore, the same contribution given by the operator R_{ν_1,ν_2} is achieved also by the map

$$R'_{1M}=\frac{1}{3}(R_{\nu_1,\nu_2}+R_{\nu_2,M-\nu_1-\nu_2-1}+R_{M-\nu_1-\nu_2-1,\nu_1}), \tag{63}$$

with the following identifications:

$$r_0^{\nu_1,\nu_2}=r_1^{\nu_2,M-\nu_1-\nu_2-1}=r_2^{M-\nu_1-\nu_2-1,\nu_1},$$

$$r_1^{\nu_1,\nu_2}=r_2^{\nu_2,M-\nu_1-\nu_2-1}=r_0^{M-\nu_1-\nu_2-1,\nu_1},$$

$$r_2^{\nu_1,\nu_2}=r_0^{\nu_2,M-\nu_1-\nu_2-1}=r_1^{M-\nu_1-\nu_2-1,\nu_1}. \tag{64}$$

The completeness constraint (17) along with Eqs. (64) lead to

$$\sum_{\nu_1,\nu_2}(r_0^{\nu_1,\nu_2})^2+\sum_{\nu_1,\nu_2}(r_1^{\nu_1,\nu_2})^2+\sum_{\nu_1,\nu_2}(r_2^{\nu_1,\nu_2})^2=3. \tag{65}$$

If we restrict our attention to the family of cloning transformations described by R_{ν_1,ν_2} we have to fulfill the constraint

$$(r_0^{\nu_1,\nu_2})^2+(r_1^{\nu_1,\nu_2})^2+(r_2^{\nu_1,\nu_2})^2=3. \tag{66}$$

Let us first consider the case $M=3k$. From Eq. (61) we can see that the representation that contributes mostly to the fidelity is that with $\overline{\nu_1}=\overline{\nu_2}=k$, because one of the three coefficients T that appear in Eq. (61) is maximized and the other two take the second possible highest value simultaneously. Therefore, we can maximize the fidelity by restricting our attention to the block labeled by $\overline{\nu_1}$ and $\overline{\nu_2}$. Moreover, since $T(M,\nu_1+1,\nu_2)$ and $T(M,\nu_1,\nu_2+1)$ have the same value for $\nu_1=\nu_2=k$, expression (61) is invariant under exchange of the coefficients $r_1^{k,k}$ and $r_2^{k,k}$. Therefore, we can set $r_1^{k,k}=r_2^{k,k}$ when we look for the optimal solution. The optimization of the contribution in Eq. (61) with $\nu_1=\nu_2=k$ corresponds to maximizing the quantity $r_0^{k,k}A+2r_1^{k,k}B$, with the constraint $(r_0^{k,k})^2+2(r_1^{k,k})^2=3$ and with $A=\sqrt{T(M,M/3,M/3)}$ and $B=\sqrt{T(M,M/3+1,M/3)}$. The solution corresponds to

$$r_0^{k,k}=\sqrt{3-2(r_1^{k,k})^2},$$

$$r_1^{k,k}=\sqrt{3}\left(\frac{A^2}{B^2}+2\right)^{-1/2}, \quad r_2^{k,k}=r_1^{k,k}, \tag{67}$$

and all other nonvanishing coefficients given by Eq. (64). The corresponding optimal map is then given by

$$R_{1M}=\frac{1}{3}(R_{M/3,M/3}+R_{M/3,M/3-1}+R_{M/3-1,M/3}), \tag{68}$$

with the nonvanishing coefficients $r_i^{\nu_1,\nu_2}$ given by Eqs. (64) and (67).

In the remaining case $M=3k+2$ the optimization argument and the final solution are the same as for $M=3k$. Here we maximize the quantity $r_0^{k,k}A+2r_1^{k,k}B$, with A

$=T(M,(M-2)/3,(M-2)/3)$ and $B=T(M,(M-2)/3,(M-2)/3+1)$. The optimal solution can be derived analogously to the previous case.

As in the case of qubits we can derive optimal maps for qutrits by maximizing the average single-particle fidelity instead of the global one, and by assuming that the operator R is supported on the symmetric subspace. As in the case of qubits we will see that the optimal maps for the average single-particle fidelity are not always the same as those derived above, where the global fidelity was maximized. Actually, in this case the contributions to the average single-particle fidelity F_{1M} are of the form

$$\begin{aligned}
&\frac{1}{9}\,\mathrm{Tr}[(|\psi_{0,0}\rangle\langle\psi_{0,0}|^{\mathbb{1}\otimes M-1}|\psi_{0,0}\rangle\langle\psi_{0,0}|)\\
&\quad\times|s_{M,\nu_1,\nu_2}\rangle\langle s_{M,\nu_1+1,\nu_2}|\otimes|0\rangle\langle 1|]\\
&\quad=\frac{1}{9}\,\frac{(M-1)!}{(M-\nu_1-\nu_2-1)!\,\nu_1!\,\nu_2!}\sqrt{\frac{(M-\nu_1-\nu_2)!\,\nu_1!\,\nu_2!}{M!}}\\
&\quad\times\sqrt{\frac{(M-\nu_1-\nu_2-1)!(\nu_1+1)!\,\nu_2!}{M!}}\\
&\quad=\frac{1}{9M}\sqrt{(M-\nu_1-\nu_2)(\nu_1+1)}\\
&\quad\equiv\frac{1}{9}\Lambda_M(M-\nu_1-\nu_2,\nu_1),
\end{aligned}\tag{69}$$

where we define

$$\begin{aligned}
\Lambda_M(p,q)&\equiv\mathrm{Tr}[(|\psi_{0,0}\rangle\langle\psi_{0,0}|^{\mathbb{1}\otimes M-1})|s_{M,p,q}\rangle\langle s_{M,p-1,q+1}|]\\
&=\frac{1}{M}\sqrt{p(q+1)}.
\end{aligned}\tag{70}$$

The arguments leading to form (59) and to constraints (65) hold also in this case. The contributions to the fidelity due to the operators R_{ν_1,ν_2} are given by

$$\begin{aligned}
F_{\nu_1,\nu_2}=\frac{1}{9}[&(r_0^{\nu_1,\nu_2})^2+(r_1^{\nu_1,\nu_2})^2+(r_2^{\nu_1,\nu_2})^2\\
&+2r_0^{\nu_1,\nu_2}r_1^{\nu_1,\nu_2}\Lambda_M(M-\nu_1-\nu_2,\nu_1)\\
&+2r_0^{\nu_1,\nu_2}r_2^{\nu_1,\nu_2}\Lambda_M(M-\nu_1-\nu_2,\nu_2)\\
&+2r_1^{\nu_1,\nu_2}r_2^{\nu_1,\nu_2}\Lambda_M(\nu_1+1,\nu_2)].
\end{aligned}\tag{71}$$

As discussed above, the optimal cloning map corresponds to optimizing the coefficients r_i for the block R_{ν_1,ν_2} that gives the maximum contribution (71). In the case $M=3k+1$, all the three terms Λ_M that appear in expression (71) are optimized at the same time for $\nu_1=\nu_2=k$, and therefore the optimal map has the same form as that found by maximizing the global fidelity. The fidelity F_{1M} in this case takes the explicit form

$$F_M=\frac{1}{3}\left(1+2\frac{M+2}{3M}\right).\tag{72}$$

Let us now consider the case of $M=3k$. By looking at Eq. (71) we can see that the maximum contribution to the fidelity corresponds to $\nu_1=\nu_2=k$, because one of the three coefficients Λ_M is optimized and the other two take the second possible highest value simultaneously. Moreover, since $\Lambda_M(M-\nu_1-\nu_2,\nu_1)$ and $\Lambda_M(M-\nu_1-\nu_2,\nu_2)$ have the same value for $\nu_1=\nu_2=k$, expression (71) being the optimal solution corresponds to $r_1^{k,k}=r_2^{k,k}$. Therefore, maximum contribution (71) corresponds to maximizing the quantity

$$2r_0^{k,k}r_1^{k,k}\Lambda_A+(r_1^{k,k})^2\Lambda_B,\tag{73}$$

with the constraint $(r_0^{k,k})^2+2(r_1^{k,k})^2=3$ and with $\Lambda_A=\sqrt{M(M+3)}/3M$ and $\Lambda_B=(M+3)/3M$. The optimal solution corresponds to

$$\begin{aligned}
r_0^{k,k}&=\sqrt{3-2(r_1^{k,k})^2},\\
r_1^{k,k}&=\frac{\sqrt{3}}{2}\sqrt{1+\frac{\Lambda_B}{\sqrt{\Lambda_B^2+8\Lambda_A^2}}},\quad r_2^{k,k}=r_1^{k,k}.
\end{aligned}\tag{74}$$

The optimal map has form (68), with the values of the coefficients r_i for eack block fixed according to Eqs. (64) and (74).

In the remaining case of $M=3k+2$ the optimization argument and the final solution are the same as in the $M=3k$ case, with $\Lambda_A=\sqrt{(M+4)(M+1)}/3M$ and $\Lambda_B=(M+1)/3M$. Notice that the optimal map coincide with the case $1\to3$ derived in Ref. [13].

We want to point out that the average single-particle fidelity in the limit $M\to\infty$ coincides with the fidelity of optimal double-phase estimation for a qutrit in state (55) [17].

V. DISCUSSION

In this paper we derived from first principles the optimal quantum cloning transformations that maximize the fidelity averaged uniformly over all equatorial qubit states. We have seen that such averaged form for the fidelity automatically leads to the optimal phase-covariant cloning. We have then derived the optimal $N\to M$ cloning transformation using the method [13] designed for group-covariant cloning. We have also considered phase-covariant cloning for qutrits, and derived the $1\to M$ optimal cloning maps. From our analytical results one can see that the fidelities are always larger than those obtained for the universal cloning [8]. Moreover, the fidelity for the qutrit cloning is smaller than the corresponding one for the qubit. We also found that the form of the optimal cloning maps depends on the criterion adopted to assess the quality of the transformation. Actually, we showed that the maximization of the global fidelity and the maximization of the average single particle fidelity in general lead to different solutions.

We want now to emphasize that the general analysis performed in Sec. II for optimal phase-covariant cloning would

be exactly the same for any smaller discrete phase-covariance group, such as for example the discrete group Z_4 of $\pi/2$-rotations that is employed in the BB84 cryptographic scheme [2]. Moreover, since the averaged fidelity is the same as that of the single state whose group orbit generates all possible input states, the only feature that can depend on the particular group in the following analysis is the irreducibility of the representation $U_\phi^{\otimes M} \otimes (U_\phi^{\otimes N})^*$. This is the same for the full rotation group $U(1)$ and for its subgroup Z_4 for $N=1$ and $M \leqslant 2$, whereas one may expect a slight improvement of fidelities for larger N and M.

Finally, we want to stress that the method used in the present paper could be easily generalized to any quality criteria—also called *cost function*—different from the averaged fidelity in Eq. (2) and the single-particle average fidelity (42). As a matter of fact, the averaging of the cost function will always lead to a an optimal cloning that is covariant, as long as the cost function is linear in the cloning CP map. However, it will not be necessarily true that the optimal cloning map will have output state in the symmetric tensor-product Hilbert space. Actually, also in the case of the single-particle fidelity we found the optimal maps starting from the assumption that the output state is supported on the symmetric subspace.

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