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Time evolution of an anharmonic oscillator interacting with a squeezed bath

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Abstract. The evolution of a single mode of the electromagnetic field interacting with a squeezed bath in a Kerr medium is considered. The solution of the corresponding master equation is given numerically. It is argued that the creation of a superposition state (Schrödinger's cat) is better achieved in the presence of a squeezed reservoir than of a thermal one.

1. Introduction

The possibility of detecting quantum interference effects between macroscopically distinguishable states—eigenstates of a variable corresponding to macroscopically different eigenvalues—has received much attention in the last few years [1–10]. Such interest is motivated by one of the fundamental problems in quantum mechanics: to determine whether quantum-mechanical features can be observed in macroscopic objects [11]. Unfortunately, the generation and detection of a macroscopic superposition is very difficult, due to the unavoidable coupling with the environment and the consequent dissipation [12]. The severity with which dissipation destroys the quantum coherence at a macroscopic level has been discussed by several authors [13–19].

Since the original proposal by Yurke and Stoler [1], who have shown that a coherent state propagating through a Kerr medium evolves, under suitable conditions, into a quantum superposition of two coherent states which are 180° out of phase with each other, many attempts have been made to preserve this superposition with detectable effects such as, for example, the interference fringes at the output of a homodyne detector. Mecozzi and Tombesi [3–5] have considered a dissipation model corresponding to a beamsplitter with a squeezed vacuum injected into the unused port. They have shown that the interference pattern may be preserved if the input light is squeezed in a suitable quadrature. As a model for phase-sensitive measurements, Kennedy and Walls [6] suggested the use of a squeezed bath in place of the thermal one, showing a substantial improvement of the quadrature-phase sensitivity. Along these lines, Bužek *et al* [10] have recently studied

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the phase properties of superpositions of coherent states evolving in a squeezed (phase-sensitive) amplifier, showing that differently from the phase-insensitive case [19], the phase distribution of the Schrödinger-cat input state can be preserved for long times.

The above models either suffer a parametrically imposed dynamics for the loss (beamsplitter models [1, 3–5]), or do not consider the nonlinear coupling generating the superposition state, which is just treated as an initial condition [6, 10]. Daniel and Milburn [19] have shown that the generation of macroscopically distinguishable quantum states in a Kerr medium is completely inhibited by the unavoidable presence of loss or dissipation into the medium. We are thus aware that the difficulty of generating these macroscopic states in such a medium lies with the technological possibility of obtaining materials with very high nonlinearity with respect to dissipation; until then it appears useless to study this topic further. Despite this, we shall show that a 'quasi-superposition' of macroscopic quantum states, with interference fringes still surviving, could be obtained when one considers a model for a phase-sensitive loss. We shall show that, even for not very high values of the ratio of nonlinearity to dissipation, the quasi-superposition is possible when the loss is modelled by a squeezed bath. For this reason we consider a model in which both the anharmonic Hamiltonian and the interaction with a squeezed bath are taken into account, thus allowing a test of the effect of squeezed fluctuations on the generation of a superposition state starting from a single coherent one.

This paper is organized as follows. In section 2 we introduce the model and discuss the corresponding master equation. In section 3 we present the numerical integration of the master equation and the results of the numerical analysis, showing the efficiency of a squeezed bath in the creation of a 'quasi-superposition' state. In section 4 we conclude and summarize the results.

2. The model and the master equation

We consider one single mode of the electromagnetic field at frequency ω travelling in a Kerr medium coupled to a squeezed bath, namely a reservoir of oscillators whose fluctuations are squeezed [6], which models the phase-sensitive loss. The total Hamiltonian is given by

$$H = H_S + H_I + H_B \tag{1}$$

where H_S is the free Kerr Hamiltonian of the field mode

$$H_S = \omega(a^\dagger a) + \Omega(a^\dagger a)^2 \tag{2}$$

H_B is the free Hamiltonian of the bath, and H_I is the oscillator–reservoir interaction Hamiltonian, which in the rotating wave approximation has the form

$$H_I = a^\dagger \hat{\Gamma} + a\hat{\Gamma}^\dagger\,. \tag{3}$$

In equations (2) and (3) $a^\dagger$ and a are the boson creation and annihilation operators of the mode, while $\hat{\Gamma}^\dagger$ and $\hat{\Gamma}$ are bath operators. The bath is squeezed and Markovian, namely the correlation functions of the operators $\hat{\Gamma}^\dagger$ and $\hat{\Gamma}$ are given by [20]

$$\begin{aligned} &\langle\hat{\Gamma}^\dagger(t)\hat{\Gamma}(t')\rangle = 2\gamma N\delta(t-t') \qquad \langle\hat{\Gamma}(t)\hat{\Gamma}^\dagger(t')\rangle = 2\gamma(N+1)\delta(t-t') \\ &\langle\hat{\Gamma}(t)\hat{\Gamma}(t')\rangle = 2\gamma M\,\mathrm{e}^{-2\mathrm{i}\omega t}\delta(t-t') \qquad \langle\hat{\Gamma}^\dagger(t)\hat{\Gamma}^\dagger(t')\rangle = 2\gamma M^*\,\mathrm{e}^{2\mathrm{i}\omega t}\delta(t-t')\,. \end{aligned} \tag{4}$$

In equations (4) γ is the damping constant (determined by the coupling between the oscillator and the bath), N is a real parameter, which reduces to the mean number of thermal photons when the bath is not squeezed, and M is the squeezing complex parameter ($M = |M|\,\mathrm{e}^{\mathrm{i}\psi}$) satisfying the relation

$$|M| \leqslant \sqrt{N(N+1)}\,. \tag{5}$$

The parameters N and M measure the strength of the correlations of the bath degrees of freedom: for $|M|^2 = N(N+1)$ the squeezing is maximum, whereas for $|M| = 0$ the reservoir reduces to the customary thermal one.

From equations (1)–(4) one can derive the following master equation for the reduced density matrix of the field mode in the interaction picture [20]:

$$\frac{\mathrm{d}\hat{\rho}}{\mathrm{d}t} = -\mathrm{i}\Omega[(a^\dagger a)^2, \hat{\rho}] + \gamma(N+1)(2a\hat{\rho}a^\dagger - a^\dagger a\hat{\rho} - \hat{\rho}a^\dagger a) + \gamma N(2a^\dagger\hat{\rho}a - aa^\dagger\hat{\rho} - \hat{\rho}aa^\dagger)$$
$$-\gamma M(2a^\dagger\hat{\rho}a^\dagger - a^\dagger a^\dagger\hat{\rho} - \hat{\rho}a^\dagger a^\dagger) - \gamma M^*(2a\hat{\rho}a - aa\hat{\rho} - \hat{\rho}aa)\,. \tag{6}$$

Equation (6) is solved numerically in the next section. The results are shown in terms of the Q-function

$$Q(\alpha, \alpha^*, t) = \langle\alpha|\hat{\rho}(t)|\alpha\rangle \tag{7}$$

which is the (positive-definite) probability density for the anti-normally ordered moments of the annihilation and creation operators, and in terms of the Wigner function, which is defined by

$$W(\alpha, \alpha^*, t) = \int \frac{\mathrm{d}^2\lambda}{\pi^2}\, \mathrm{e}^{-\lambda\alpha^*+\lambda^*\alpha} \left\{\hat{\rho}(t)\, \mathrm{e}^{\lambda a^\dagger - \lambda^* a}\right\}. \tag{8}$$

Using standard methods [13], it is possible to convert the master equation (6) into the Fokker–Planck-type equation for the Q-function

$$\frac{\partial Q}{\partial t} = -\mathrm{i}\Omega\left[\alpha^*(1+2|\alpha|^2)\frac{\partial Q}{\partial\alpha^*} - \alpha(1+2|\alpha|^2)\frac{\partial Q}{\partial\alpha} + (\alpha^*)^2\frac{\partial^2 Q}{\partial\alpha^{*2}} - \alpha^2\frac{\partial^2 Q}{\partial\alpha^2}\right]$$
$$+\gamma\left(\frac{\partial}{\partial\alpha}\alpha + \frac{\partial}{\partial\alpha^*}\alpha^*\right)Q + 2\gamma(N+1)\frac{\partial^2 Q}{\partial\alpha\partial\alpha^*} + \gamma M\frac{\partial^2 Q}{\partial\alpha^2} + \gamma M^*\frac{\partial^2 Q}{\partial\alpha^{*2}}\,. \tag{9}$$

Equation (9) has been solved for $\Omega = 0$ [6] and for $M = 0$ [19]. For Ω and M both non-vanishing, however, an analytic solution is not available, whereas numerical integration has to be carried out carefully, because of computation instabilities [21].

In absence of dissipation an initial coherent state

$$|\alpha_0\rangle = \exp\left(-\frac{|\alpha_0|^2}{2}\right)\sum_{n=0}^{\infty}\frac{\alpha_0^n}{\sqrt{n!}}|n\rangle \tag{10}$$

evolves towards the following superposition of coherent states at $t = \pi/2\Omega$

$$|\phi\rangle = \frac{1}{\sqrt{2}}\left(\mathrm{e}^{-\mathrm{i}\pi/4}|\alpha_0\rangle + \mathrm{e}^{\mathrm{i}\pi/4}|-\alpha_0\rangle\right). \tag{11}$$

as was shown by Yurke and Stoler [1]. The Q-function of the $t = 0$ coherent state is the Gaussian

$$Q(\alpha, \alpha^*, 0) = \exp(-|\alpha - \alpha_0|^2) \tag{12}$$

whereas for $t = \pi/2\Omega$ it corresponds approximatively to two Gaussian peaks in the complex plane centred at α_0 and $-\alpha_0$, with interference fringes between them [1]. The two peaks are 'macroscopically distinguishable' for $|\alpha_0| \gg 1$. The evolution is periodic with period $T = 2\pi/\Omega$. Of course, at shorter times the superposition involves more coherent states [1]. In the presence of dissipation the state (at times $t > 0$) is no longer pure [19]. However, we shall show that with the master equation (6) a superposition of states can be achieved which corresponds to a quasi-superposition mixed state, with interfering features still surviving.

We will look for signatures of such states from peaks in the complex plane approximately separated by a distance $2|\alpha_0|$, reminiscent of the state (11), and oscillations in the marginal distribution.

3. Numerical results

We have studied the time evolution of the density matrix by integrating numerically the master equation (6) for truncated Hilbert space dimension $d = 128$ (typically a power of 2, in order to take advantage of the fast Fourier transform algorithm for reconstructing the Q-function and the Wigner function). The integration time-step has to be carefully tuned as a function of Ω and α_0 (here typically $\gamma\Delta t \simeq 10^{-5}$ for a standard fourth-order Runge–Kutta

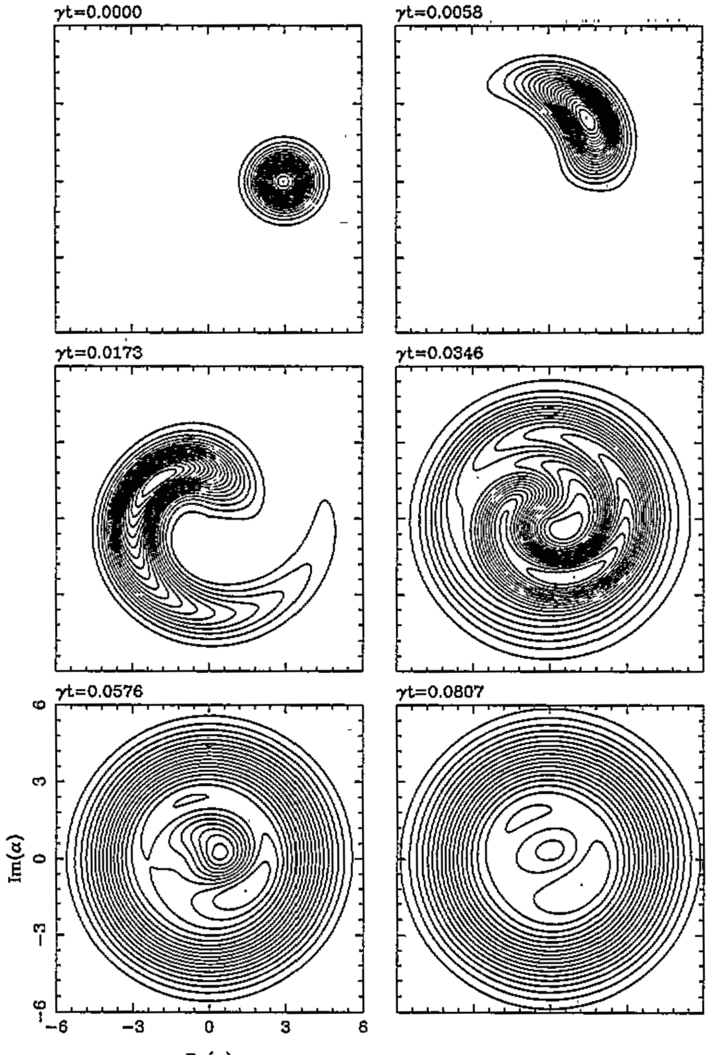

Figure 1. Contour plots of the time evolved Q-function according to equation (6). Here $N = 30$, $|M| = \sqrt{N(N+1)}$, $\psi = 0$, $\alpha_0 = 3$, and $\Omega/\gamma = 10$.

routine). Numerical accuracy is checked through normalization of $\hat{\rho}$, Q and W, positivity of Q, and reality of the diagonal elements of $\hat{\rho}$. As a test, the results from analytical solutions for $\gamma = 0$ [1, 2] and for $\Omega = 0$ [6] have been recovered up to the seventh digit.

In figure 1 the Q-function from the master equation (6) for maximally squeezed bath $[|M| = \sqrt{N(N+1)}]$ is given for $\Omega/\gamma = 10$, $N = 30$, and squeezing phase $\psi = 0$. At $\gamma t = 0.0576$ (well before $\Omega t = \pi/2$, namely before the creation time of the state (11) in the undamped case) two peaks are clearly visible in the structure of the Q-function. The two peaks are almost symmetrical with respect to the origin of the phase space. Of course, due to dissipation, they are not strictly coherent, but they are still visible, even for subsequent times and disappear at larger times. This result can be compared with that of Daniel and Milburn [19] at the same time, but in the absence of the squeezed bath.

In figure 2 the effect of a purely thermal bath on the generation of the superposition state is shown for comparison with figure 1 for the same values of the parameters. In this case there are no peaks in the Q-function, which now exhibits classical behaviour [17]: this is the usual effect of dissipation [14].

In figure 3, for completeness, we show the time evolution of the Wigner function at the same times as in figure 1: the two peaks still show up, but now some interference features arise between the two component states, as expected because the Wigner function is more sensitive to quantum features.

The effect is further confirmed by the behaviour of the quadrature distributions $P(x_\phi)$,

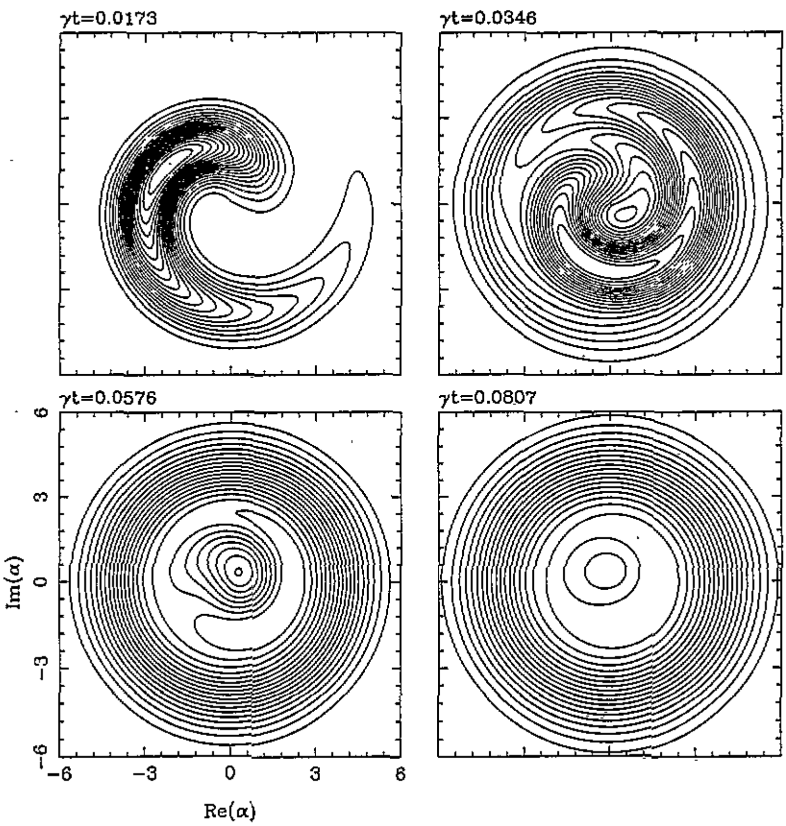

Figure 2. The same as in figure 1, but for $M = 0$.

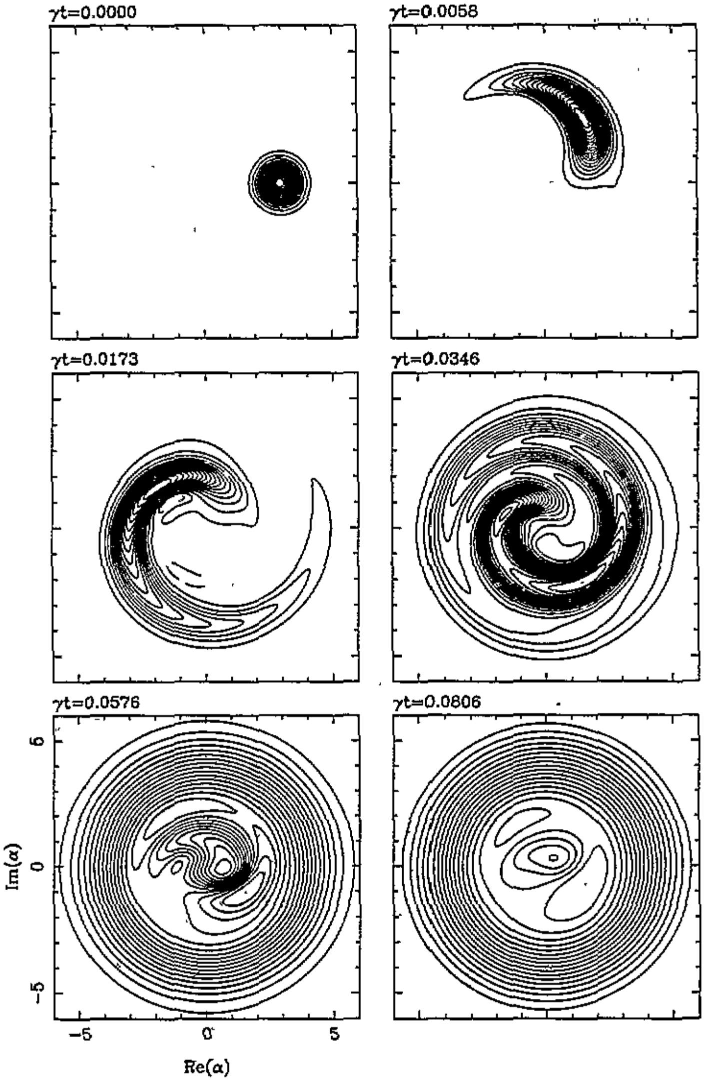

Figure 3. The Wigner function corresponding to plots in figure 1.

defined by

$$P(x_\phi) = \langle x_\phi|\hat{\rho}|x_\phi\rangle \tag{13}$$

with

$$\hat{x}_\phi = \frac{1}{\sqrt{2}}\left(e^{i\phi}a + e^{-i\phi}a^\dagger\right) \tag{14}$$

with $|x_\phi\rangle$ being the eigenstate of $\hat{x}_\phi$ with eigenvalue x_ϕ. The probability $P(x_\phi)$ is plotted in figure 4 at $\gamma t = 0.0576$ for $\phi = \pi/4$ and $\phi = 3\pi/4$. For $\phi = 3\pi/4$—along the direction which joins the two peaks of the Wigner function—the probability $P(x_\phi)$ exhibits two peaks resulting from marginal integration in the complex plane. On the other hand, along the orthogonal direction at $\phi = \pi/4$, a pattern reminiscent of interference in phase space [22] between the two originally coherent components is visible. We have also computed

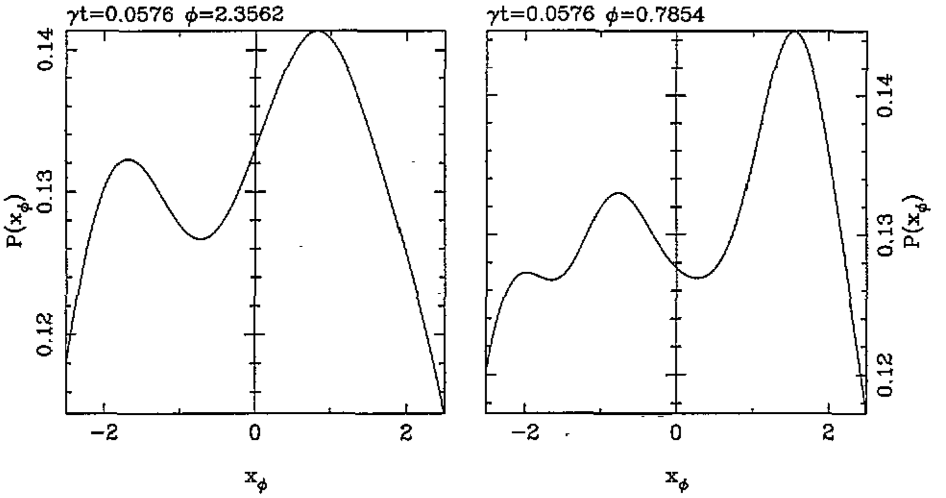

Figure 4. Plot of the marginal distribution $P(x_\phi)$ at $\gamma t = 0.0576$ for $\phi = 3\pi/4$ and $\phi = \pi/4$.

the degree of mixing $S = 1 - \mathrm{Tr}(\hat{\rho}^2)$: its behaviour shows a rapid increase with time, achieving the asymptotic value $S = 0.96$. Thus, the above coherence features still survive notwithstanding the large mixing.

It is interesting to note that the time at which we have the formation of the two peaks is approximately the same as the one at which—in the absence of dissipation—five peaks show up. To make this comparison we have to consider the scaled time $\tau = \Omega t$ instead of γt. In figure 5, for comparison, we show six different time evolutions of the Q function, starting again from a single coherent state, at the same time $\tau = \pi/5$. In figure 5(a) we show the pure Kerr effect, in the absence of dissipation: a coherent superposition of five coherent states is clearly visible. In figure 5(b) the same evolution is considered, but for a thermal bath, and the peaks are completely destroyed. On the other hand, figures 5(c)–(f) refer to the above-mentioned squeezed bath, for different values of the phase ψ of the squeezing parameter. It is easily seen that depending upon the choice of ψ, one can select which two peaks (among the five original ones) survive (see also [23] for a similar effect due to a phase-sensitive amplifier). This shows clearly that dissipation rapidly washes out the possibility of the creation of the superposition state, but also that by squeezing the degrees of freedom of the bath it is still possible to produce 'quasi-superposition' states which show coherence. It is not surprising that the phase of the squeezing parameter rules the effect of the squeezed bath in preserving the production of quasi-superposition states, due to the introduction of a privileged direction [24]. It should be noted, however, that such an effect of the squeezed reservoir is limited to short times: at greater times τ the coherent effects disappear.

As regards the other parameters, the overall scenario is not substantially affected by changing the phase of the initial coherent field α_0 (though modifying the value of the phase of α_0 changes the position of the peaks in the phase space), whereas increasing the modulus of α_0 deteriorates the formation of the two peaks. On the other hand, slightly increasing Ω or N (still for a maximally squeezed bath), does not appreciably improve the visibility of the peaks, unless we use a very large Ω.

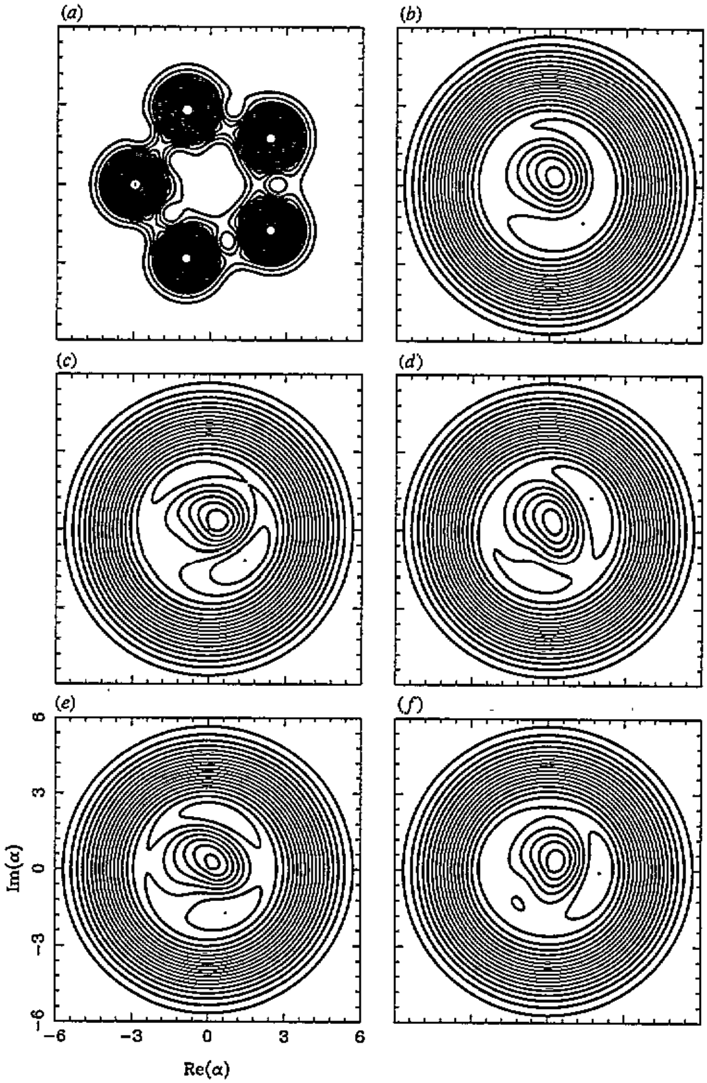

Figure 5. Comparison of different dynamical evolutions of the Q function for an initial state given by a single coherent state $|\alpha_0\rangle = |3.0\rangle$. The contour plots of the time evolved Q function according to (6) at the same time $\Omega t = \pi/5$ are shown for (a) the pure Kerr effect ($\gamma = 0$); (b) a thermal bath with $N = 30$ ($M = 0$); (c) a squeezed bath with $N = 30$, $|M| = \sqrt{N(N+1)}$, $\psi = 0$; (d) the same as (c) but with $\psi = \pi$; (e) the same as (c) but with $\psi = \pi/2$; (f) the same as (c) but with $\psi = 3\pi/2$.

4. Summary and conclusions

We have considered the anharmonic oscillator interacting with a squeezed bath. The dissipative Kerr effect models a coherent state propagating into a nonlinear medium, such as, for example, into an optical fibre. The squeezing has been considered in order to produce coherent effects in the presence of dissipation. Differently from previous works [6, 10], our model treats all effects (nonlinearity, dissipation and squeezing) in context. We have shown that squeezing the fluctuations of the bath improves the generation of 'quasi-superposition' states, which are completely forbidden in the presence of a heat bath. Our results are in agreement with those of Kennedy and Walls [6] and of Bužek *et al* [10] and complete them: not only is a squeezed bath able to preserve a macroscopic quantum superposition, but also allows its generation in the presence of nonlinear interactions. It is commonly believed that in order to produce Schrödinger cat states in a Kerr medium, high ratios Ω/γ are needed [25]. In the present work we have shown that the value $\Omega/\gamma = 10$ is sufficient. Furthermore, new semiconductor materials have been recently found to show high third-order nonlinear susceptibility and also large ratios of nonlinearity to loss [26]. This appears to give new interest to the search for the generation of superpositions of macroscopically distinguishable quantum states in Kerr media.

In order to be complete, we have to say that in the present work we have considered the production of superpositions in the presence of dissipation, whereas much care must be taken if one also considers the measurement of the state. The physical meaning of a squeezed bath is not completely understood in the present context: a suitable feedback mechanism could be envisaged that supports an *ad hoc* phase-sensitive interaction, which amplifies or attenuates fluctuations, depending on the quadrature [27]. Alternatively, it could be realized by inserting various phase-sensitive amplifiers along the line to provide a similar effect as in the linear optical fibre [28]. Notwithstanding, squeezing seems to be the only way to maintain coherent effects in the presence of noise, provided the set of values of the bath parameters belong to a narrow region of the parameter space [6].

In conclusion, some comments regarding the direction of squeezing in the bath are in order. Despite the squeezing, direction does not affect the generation of quasi-superposition states, which are only given by the nonlinearity, the survival time of coherence is naturally increased only for squeezing in the direction orthogonal to the line that joins the component states. On the other hand, the Kerr effect rotates the Q-function in the complex plane, thus making squeezing less efficient in the overall evolution. This suggests that an ideal bath should be squeezed isotropically in the plane, depending on the phase of the state itself. It is possible to write a master equation for such an isotropically squeezed bath: numerical results on these lines can be found in [29].

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