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Entanglement at distance: qubits versus continuous variables

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We consider the problem of obtaining maximally entangled photon states at distance in the presence of loss. We compare the efficiency of two different schemes in establishing N shared ebits: i) N single ebit states with the qubit encoded on polarization; ii) a single continuous variable entangled state (emode) assisted by optimal local operation and classical communication (LOCC) protocol in order to obtain a 2^N-dimensional maximally entangled state, with qubits encoded on the photon number.

1 Introduction

The production of maximally entangled state of photons at distance is a key issue in communications of quantum information, for distributed quantum computation [1], quantum teleportation [2], and quantum cryptography [3]. Unfortunately, the detrimental effect of losses is a serious problem for establishing entangled resources at distance, since any kind of non-classical state is very sensitive to the effect of loss.

If one needs to teleport N qubits from Alice to Bob, N ebits need to be shared between them, and for such purpose photons are the only practically available carriers. One can use equivalently either N single ebit with the qubit encoded on polarization, or a single continuous variable entangled state – "emode" – with qubits encoded on the photon number. Parametric downconversion allows to create both kinds of entangled states, ebits and emodes, in the low and high gain regimes, respectively.

In this paper we compare ebits and emodes in the presence of loss. In contrast to the case of a single emode, a scheme based on ebits with the qubit encoded on the polarization of single photons has the obvious advantage that the successful achievement of the ebit is automatically checked by the presence of the photon itself at both Alice and Bob sites, whereas for a single emode, this is not possible, due its vacuum component. While there is no viable method for testing the presence of the emode without destroying the entanglement, a scheme for purification of emodes in the presence of loss has been proposed in [4], however, with the disadvantage of achieving a maximally entangled state of a random set of modes, hence with the need of changing the encoding/decoding procedure each time, depending on which are the entangled modes. Therefore, in the presence of loss only the ebit allows *knowingly* successful teleportation/communication without increasing the complexity of the protocol versus N. Since emodes are more sensitive to loss for increasingly large number of photons, a way out for implementing emodes in the presence of moderate losses is to produce weakly entangled modes, then performing a LOCC operation to enhance the entanglement at the output of the lossy channel: this is the only viable method for designing a protocol based on ebits with low probability of failure.

In this paper we will compare the efficiency of ebits and emodes in establishing N maximally entangled ebits, by considering a protocol for emodes in which weakly entangled states are prepared and an optimal

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LOCC operation is performed after the loss in order to obtain a 2^N maximally entangled state. As we will see, the ebits are largely superiors to emodes in all cases.

2 The comparison

Our task is to create N ebits shared at distance between Alice and Bob in the presence of loss. N ebits are represented by N copies of a maximally entangled state belonging to the tensor-product $\mathbb{C}^2 \otimes \mathbb{C}^2$ of two-dimensional Hilbert spaces, or equivalently by a single maximally entangled state in $\mathbb{C}^{2^N} \otimes \mathbb{C}^{2^N}$. To achieve this task we consider the use of N ebits and the use of a single emode, with qubits encoded on photon polarization and photon number, respectively.

First we consider the use of N single-photon states

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|0\rangle_a|1\rangle_b + |1\rangle_b|0\rangle_b) \;, \tag{1}$$

where the subscripts a, b denote Alice and Bob sites.

The effect of loss on a single-mode state ρ is described by a completely positive map that can be written in the Kraus form [5]

$$\mathcal{L}[\rho] \doteq \sum_{n=0}^{\infty} V_n \rho V_n^\dagger \;, \tag{2}$$

with

$$V_n = \frac{(\eta^{-1}-1)^{n/2}}{\sqrt{n!}} a^n \eta^{1/2 a^\dagger a} \;, \qquad 0 \le \eta \le 1 \tag{3}$$

The physical parameter η plays the role of the energy attenuation factor, since on has $\mathrm{Tr}[\mathcal{L}[\rho]a^\dagger a] = \eta \mathrm{Tr}[\rho a^\dagger a]$. The smaller is the value of η, the larger is the effect of the loss. More generally, η gives the scaling factor of any normal-ordered operator function, namely

$$\mathcal{L}^\vee{:}f(a^\dagger, a){:} = {:}f(\eta^{1/2}a^\dagger, \eta^{1/2}a){:} \;, \tag{4}$$

where $\mathcal{L}^\vee$ denotes the dual map, which is defined through the identity

$$\mathrm{Tr}\left[\mathcal{L}^\vee[O]\rho\right] = \mathrm{Tr}\left[O\mathcal{L}[\rho]\right] \tag{5}$$

valid for any operator O.

The typical best achievable value of the loss in a optical fibers is of order 0.3 dB/km. Hence for two parties 10 km far apart the loss is 3 dB, corresponding to $\eta = 1/2$.

Each mode is affected by the effect of loss, hence from the maximally entangled state (1) one obtains the mixture

$$\mathcal{L}_a \otimes \mathcal{L}_b[|\psi\rangle\langle\psi|] = \eta|\psi\rangle\langle\psi| + (1-\eta)|0\rangle_{aa}\langle 0| \otimes |0\rangle_{bb}\langle 0| \;. \tag{6}$$

Therefore, the probability of sharing N maximally entangled ebits in the presence of loss is given by $p_b = \eta^N$.

Now we consider the second scheme, based on a single emode ("twin-beam")

$$|\chi(\lambda)\rangle = \sqrt{1-|\lambda|^2} \sum_{i=0}^{\infty} \lambda^i |i\rangle_a |i\rangle_b \;, \qquad |\lambda| < 1 \;. \tag{7}$$

When producing the state (7) by parametric down-conversion, the value of parameter λ is related to the gain G of the optical amplifier as $G = (1-|\lambda|^2)^{-1}$. Typically, one has $|\lambda| = 0.2 \div 0.75$ [6]. The state (7) is the entangled resource for the continuous variable teleportation of [7]. In a way analogous to Eq. (6), the twin-beam state that has suffered the effect of loss becomes a mixture, and here we are interested only in the component that is still a twin-beam, which is given by

$$V_0 \otimes V_0 |\chi(\lambda)\rangle\langle\chi(\lambda)| V_0^\dagger \otimes V_0^\dagger = \frac{1-|\lambda|^2}{1-\eta|\lambda|^2} |\chi(\eta^{1/2}\lambda)\rangle\langle\chi(\eta^{1/2}\lambda)| \ . \tag{8}$$

We rewrite the state in Eq. (7) damaged by the loss as follows

$$\mathcal{L}_a \otimes \mathcal{L}_b[|\chi(\lambda)\rangle\langle\chi(\lambda)|] = q\, |\chi(\eta^{1/2}\lambda)\rangle\langle\chi(\eta^{1/2}\lambda)| + \sigma \ , \tag{9}$$

with

$$q = \frac{1-|\lambda|^2}{1-\eta|\lambda|^2} \ , \tag{10}$$

and σ is a positive operator with $\mathrm{Tr}[\sigma] = 1 - q$. The value q gives the probability that the twin-beam state survives the loss, a part from the gain rescaling $\lambda \to \eta^{1/2}\lambda$.

Our task is now to achieve the maximally entangled state

$$|\phi\rangle = \frac{1}{\sqrt{M}} \sum_{i=0}^{M-1} |i\rangle_a |i\rangle_b \ . \tag{11}$$

For this purpose we perform a LOCC transformation on the state $|\chi(\eta^{1/2}\lambda)\rangle$. From Vidal's theorem [8], we know that the maximal probability p^* of obtaining the state $|\phi\rangle$ from $|\chi\rangle$ by means of a LOCC is given in terms of the Schmidt coefficients $\{\phi_i\}$ and $\{\chi_i\}$ of the states by

$$p^* = \min_i \frac{\sum_{n=i}^\infty \chi_n^2}{\sum_{n=i}^\infty \phi_n^2} \ . \tag{12}$$

In our case one has

$$p^* = \min_{i\in[0,M-1]} \frac{M(\eta|\lambda|^2)^i}{M-i} \le M(\eta|\lambda|^2)^{(M-1)} \equiv p' \ . \tag{13}$$

Moreover, one can easily show that for $|\lambda| \le (\eta e)^{-1/2}$ one has $p^* = p'$. Hence, the overall probability p_m of obtaining the maximally entangled state $|\phi\rangle$ using a twin-beam $|\chi(\lambda)\rangle$ in the presence of loss by means of an optimal LOCC transformation is given by

$$p_m = q\, p^* = \frac{1-|\lambda|^2}{1-\eta|\lambda|^2}\, p^* \ . \tag{14}$$

In order to compare N ebits versus a single emode one takes $M = 2^N$ for the state (11) and compares the probabilities p_b (ebits) with p_m (emodes). Some numerical results are shown in Fig. 1, where the probabilities p_b (circles) and p_m (triangles) are reported for different values of η and λ. The comparison is dramatic: qubits are much more efficient than emodes for any realistic value of the gain parameter λ and loss η.

The much greater efficiency of ebits versus emodes can be inspected analytically as follows. We define the ratio $r \equiv p_b/p'$ and we have the chain of inequalities

$$r = \frac{p_b}{p'} \le \frac{p_b}{qp'} \le \frac{p_b}{qp^*} = \frac{p_b}{p_m} \ . \tag{15}$$

From Eq. (15) it is clear that $r > 1$ implies $p_b > p_m$, namely the scheme based on N ebits is more efficient than that based on a single emode. The ratio r writes as

$$r = \left(\frac{\eta}{2}\right)^N \left(\frac{1}{\eta|\lambda|^2}\right)^{2^N-1} = \exp\{(1-2^N)\ln(\eta|\lambda|^2) + N\ln(\eta/2)\} \ . \tag{16}$$

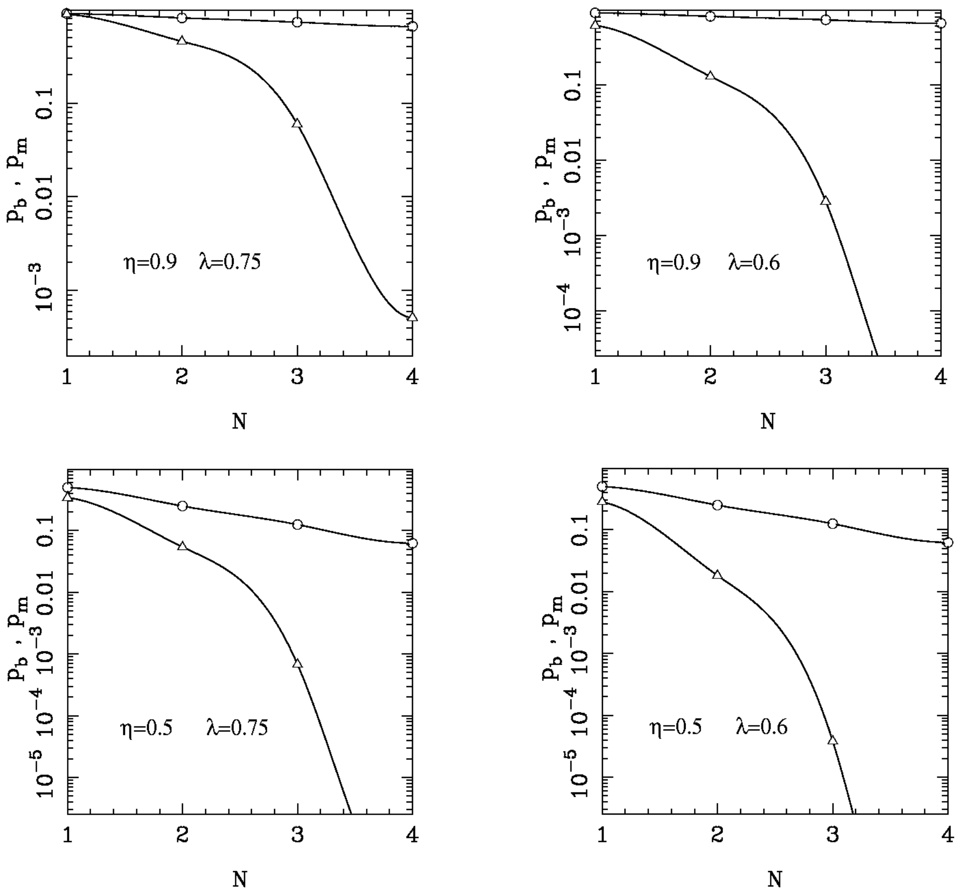

Fig. 1 Probability of successfully obtaining N shared ebits in the presence of loss using: i) (circles) N single ebit states with the qubit encoded on polarization; ii) (triangles) a single continuous variable entangled state (emode) assisted by optimal local operation and classical communication (LOCC) protocol in order to obtain a 2^N-dimensional maximally entangled state, with qubits encoded on the photon number. We considered different values of the loss η and of the gain parameter of emodes $|\lambda|$ (twin-beams in Eq. (7)).

The expression of r shows that for sufficiently large N one has $r > 1$, and the use of N ebits becomes rapidly much more efficient than using a single emode. In addition, we want to emphasize again that in the presence of loss only the ebit allows *knowingly* successful teleportation/communication without increasing the complexity of the protocol versus N.

3 Conclusions

We have considered the problem of obtaining maximally entangled photon states at distance in the presence of loss, and compared the efficiency in establishing N shared ebits by using N single ebit states – with the qubit encoded on polarization – and a single continuous variable entangled state (which we called "emode") assisted by an optimal LOCC protocol in order to obtain a 2^N-dimensional maximally entangled state, with qubits encoded on the photon number. We have shown the dramatic superiority of N ebits versus a single emode, besides the fact that only the ebit allows *knowingly* successful teleportation/communication. We have not considered the possibility of purification schemes for emodes. However, we emphasize again that the only proposed scheme [4] has the disadvantage of achieving a maximally entangled state of a random set of modes, with the need of encoding/decoding procedures whose complexity is increasing versus N. We conclude that a fruitful use of twin beams in quantum information technology at distance completely relies on the possibility of practical purification schemes, which need to be properly designed in a way which is suitable to the particular entanglement-based protocol of interest.

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