# There Exist Nonorthogonal Quantum Measurements that are Perfectly Repeatable

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We show that, contrary to the widespread belief, in quantum mechanics repeatable measurements are not necessarily described by orthogonal projectors—the customary paradigm of observable. Nonorthogonal repeatability, however, occurs only for infinite dimensions. We also show that, when a nonorthogonal repeatable measurement is performed, the measured system retains some "memory" of the number of times that the measurement has been performed.

The main paradigm of quantum mechanics is the unavoidable disturbance of the measurement on the measured system. This has obvious disruptive consequences for an objective interpretation [1] of the physical experiment. In order to retain some objectivity, one supposes the feasibility of some "canonical" measurements that can be actually regarded as the process of "seizing" a property or quantity possessed by the system independently of the measurement, thus assuming the existence of perfect measurements that satisfy the *repeatability hypothesis* formulated by von Neumann [2]: If a physical quantity is measured twice in succession in a system, then we get the same value each time. From this hypothesis, it is then concluded that the state after the measurement is the eigenvector corresponding to the measurement outcome as the eigenvalue. In the conventional approach to quantum measurements, this is the content of the so-called von Neumann "collapse" postulate, which von Neumann posed as a kind of universal law, based on the Compton and Simmons experiment.

Actually, as von Neumann himself admitted, for a degenerate observable there are many different ways of satisfying the repeatability hypothesis, with the state after the measurement given by any mixture of eigenstates corresponding to the same outcome. The concept of degenerate observable is crucial at the foundational level (i.e., to define local measurements on many particles) and, in order to retain repeatability, further physical hypotheses are needed to characterize a canonical measurement. Such an additional hypothesis was introduced by Lüders [3] in the form of a requirement of *least disturbance*, leading to the von Neumann-Lüders *projection postulate*, according to which the measurement of a discrete observable projects the state orthogonally on the eigenspace corresponding to the outcome.

In the modern formulation of quantum measurement based on "instruments" by Davies and Lewis [4], repeatable measurements are just a special type of measurement, and generally the state change after the measurement—the so-called "state reduction"—is not presupposed. However, for continuous spectrum observables—such as position and momentum—no projection postulate can apply, since the eigenvectors are not normalizable, whence they do not correspond to any physical state (in their place the notion of "posterior states" determined by the instrument was introduced by Ozawa [5]). As conjectured by Davies and Lewis [4] and then proven by Ozawa in full generality [5,6], for a continuous spectrum no instrument can satisfy a repeatability hypothesis, even in its weakest conceivable form.

In the above scenario, the orthogonal projection generally remained synonymous of repeatability [7]: However, as we will show here, repeatable measurements are not necessarily associated with orthogonal projectors. In the following, we will completely characterize all nonorthogonal quantum measurements which are perfectly repeatable, also providing explicit examples. We will then show that, due to their particular structure, nonorthogonal repeatable measurements somehow "memorize" on the system how many times the measurement has been performed.

Because of the mentioned impossibility theorem for a continuous spectrum [5,6], we will consider a measurement with discrete sample space $\mathcal{H} = \{1, 2, 3, \ldots\}$ as a denumerable collection of compatible elementary events, hereafter referred to as "outcomes." For our purpose, we can also restrict the attention to the case of pure measurements, i.e., which keep an input pure state as pure: The generalization to mixing measurement is straightforward.

A pure measurement with discrete sample space $\mathcal{H}$ on a quantum system is fully described by a set of contractions $\{M_e\}$ on the Hilbert space $\mathsf{H}$ of the system for each measurement outcome $e \in \mathcal{H}$ ("contraction" means that the operator norm is bounded as $\|M_e\| \leq 1$. We remind that the squared norm $\|A\|^2$ of an operator $A$ is defined as the supremum of $\langle\psi|A^\dagger A|\psi\rangle$ over all normalized vectors $|\psi\rangle \in \mathsf{H}$). The state after the measurement with outcome $e$ is given by

$$|\psi\rangle \mapsto |\psi\rangle_e = \frac{M_e|\psi\rangle}{\|M_e|\psi\rangle\|}, \qquad (1)$$

and occurs with probability given by the Born rule,

$$p(e) = \|M_e|\psi\rangle\|^2. \qquad (2)$$

Normalization of probabilities implies the completeness

$$\sum_{e \in \mathcal{H}} M_e^\dagger M_e = I. \qquad (3)$$

We now want to determine the most general conditions under which the measurement is perfectly repeatable. This means that the conditional probability $p(f|e)$ of obtaining the outcome $f$ at a repetition of the measurement, given the previous outcome was $e$, is the Kronecker delta $p(f|e) = \delta_{ef}$. In simple words, once any outcome is obtained, all repetitions will give the same result. In terms of the state reduction (1), we have

$$p(f|e) = \frac{\|M_f M_e|\psi\rangle\|^2}{\|M_e|\psi\rangle\|^2} = \delta_{ef} \qquad \forall \, |\psi\rangle \in \mathsf{H}, \qquad (4)$$
$$\forall \, e, f \in \mathcal{H},$$

and, in particular, for $e = f$, Eq. (4) simplifies as

$$\|M_e^2|\psi\rangle\| = \|M_e|\psi\rangle\|. \qquad (5)$$

We will now prove three lemmas, which provide a thorough mathematical characterization of repeatable measurements, and will be helpful in reconstructing the general form of the measurement contractions. The reader who is not interested in the mathematical treatment and is seeking an intuitive understanding can jump directly to the examples in Eqs. (14) and (23), and check the repeatability condition (4). For the reader who is also interested in the mathematics, only the basic theory of operators on the Hilbert spaces will be needed.

Before stating the lemmas, we will introduce some notation. The symbol $\mathrm{Ker}(O)$ will denote the *kernel* of the operator $O$, namely, the space of all vectors on which $O$ is null. The symbol $\mathrm{Supp}(O)$ denotes the *support* of $O$, i.e., the orthogonal complement of the kernel, which by definition is a subspace. Finally, $\mathrm{Rng}(O)$ denotes the *range* of $O$, i.e., the space of all output vectors $|\phi\rangle = O|\psi\rangle$ for any $|\psi\rangle$ in the Hilbert space $\mathsf{H}$. Since any contraction is bounded and defined on all $\mathsf{H}$, its kernel and range are both closed subspaces of $\mathsf{H}$, whence in the following we will use their respective symbols to denote their closures. Also, we will use the symbol $P_\mathsf{K}$ to denote the orthogonal projector on a subspace $\mathsf{K} \subseteq \mathsf{H}$.

*Lemma 1.*—With the normalization condition (3), the repeatability condition (4) is equivalent to

$$M_e^\dagger M_e|_{\mathrm{Rng}(M_e)} \equiv P_{\mathrm{Rng}(M_e)}. \qquad (6)$$

*Moreover, one has $M_f M_e = 0$ for $e \neq f$.*

Proof.—That repeatability implies Eq. (6) follows from identity (5). In fact, by posing $|\varphi\rangle = M_e|\psi\rangle$, one has $\|M_e|\varphi\rangle\|^2 = \||\varphi\rangle\|^2$ for any $|\varphi\rangle \in \mathrm{Rng}(M_e)$, which implies that $M_e^\dagger M_e$ is the identity when restricted to $\mathrm{Rng}(M_e)$. To prove the converse implication, we first see that Eq. (6) implies that $\|M_e^2|\psi\rangle\|^2 = \|M_e|\psi\rangle\|^2$ for $|\psi\rangle \in \mathrm{Rng}(M_e)$. Then, by applying the normalization condition

(3) and identity (6), one has

$$M_e^\dagger M_e|\psi\rangle = |\psi\rangle \equiv \sum_f M_f^\dagger M_f|\psi\rangle, \qquad (7)$$

which implies that $\sum_{f \neq e} M_f^\dagger M_f|\psi\rangle = 0$, and since the operators $M_f^\dagger M_f$ are all positive, one has $M_f^\dagger M_f|\psi\rangle = 0 \, \forall \, f \neq e$, then the only possibility is that $|\psi\rangle \in \mathrm{Ker}(M_f)$ for all $f \neq e$ [due to the inclusion $\mathrm{Rng}(O) \subseteq \mathrm{Ker}(O^\dagger)^\perp$, which holds for any operator $O$]. Therefore, one has $M_f M_e|\varphi\rangle = 0$ for all $|\varphi\rangle \in \mathsf{H}$.

An equivalent lemma is the following.

*Lemma 2.*—With the normalization condition (3), the repeatability condition (4) is equivalent to

$$\mathrm{Rng}(M_e) \subseteq \mathrm{Ker}(M_f), \qquad (8)$$

*for all $f \neq e$.*

Proof.—That repeatability implies Eq. (8) is an immediate consequence of the previous lemma. To prove the converse statement, consider a vector $|\psi\rangle \in \mathrm{Rng}(M_e)$. Now, Eqs. (3) and (8) imply Eq. (7). This means that $M_e^\dagger M_e$ acts as the identity on $\mathrm{Rng}(M_e)$, namely Eq. (6), which according to the previous lemma is equivalent to repeatability.

Finally, we have a necessary but not sufficient condition expressed by the following lemma.

*Lemma 3.*—With the normalization condition (3), the repeatability condition (4) implies that $\forall \, e, f \in \mathcal{H}$, $e \neq f$

$$\mathrm{Rng}(M_e) \subseteq \mathrm{Supp}(M_e), \qquad \mathrm{Rng}(M_e) \perp \mathrm{Rng}(M_f). \qquad (9)$$

Proof.—We can decompose the Hilbert space $\mathsf{H}$ as a direct sum

$$\mathsf{H} = \mathrm{Ker}(M_e) \oplus \mathrm{Supp}(M_e) \qquad (10)$$

for all $e \in \mathcal{H}$. Now suppose by absurdum that a vector $|\psi\rangle \in \mathsf{H}$ exists such that

$$M_e|\psi\rangle = |v\rangle + |\psi'\rangle, \qquad (11)$$

with $|v\rangle \in \mathrm{Ker}(M_e)$ and $|\psi'\rangle \in \mathrm{Supp}(M_e)$. Then, since $\|M_e\| \leq 1$, using Eq. (5) we have

$$\|\psi'\|^2 \geq \|M_e\psi'\|^2 = \|M_e^2\psi\|^2 = \|M_e\psi\|^2 = \|v\|^2 + \|\psi'\|^2, \qquad (12)$$

and this is possible if and only if $|v\rangle = 0$. Therefore, we have $\mathrm{Rng}(M_e) \subseteq \mathrm{Supp}(M_e)$. This relation along with Eq. (8) gives the orthogonality between the closures of the ranges, since $\mathrm{Rng}(M_e) \subseteq \mathrm{Ker}(M_f) = \mathrm{Supp}(M_f)^\perp \subseteq \mathrm{Rng}(M_f)^\perp$.

From the last lemma, it follows that only for finite dimensional $\mathsf{H}$ we have the customary orthogonal measurement paradigm.

*Corollary 1.*—For finite dimensional $\mathsf{H}$ a measurement is repeatable iff it is orthogonal.

Proof.—For finite dimensional $\mathsf{H}$, the support and the range of any operator have the same dimension, and this

fact along with the first condition in Eq. (9) implies $\text{Rng}(M_e) \equiv \text{Supp}(M_e)$. Thus, the operators $M_e^\dagger M_e = P_e$, for $e \in \mathcal{H}$, form an orthogonal projective POVM, namely,

$$P_e P_f = \delta_{ef} P_f. \tag{13}$$

For infinite dimensional $\mathsf{H}$, on the contrary, we cannot draw the same conclusion, since a subspace can have the same (infinite) dimension of a space in which it is strictly included. In fact, it is easy to construct counterexamples of repeatable measurements, as that given in the following, which satisfy conditions (6) or (8), and do not satisfy the stronger orthogonality condition (13).

Example.—*The following set of contractions*

$$M_l = \sqrt{p_l}|l\rangle\langle 0| + \sum_{j=0}^{\infty} |n(j + 1) + l\rangle\langle nj + l|, \tag{14}$$

$$1 \leq l \leq n,$$

*with $p_l \geq 0$, $\sum_l p_l = 1$, and $|n\rangle$ a generic discrete basis for the Hilbert space, defines a perfectly repeatable pure measurement with sample space $\mathcal{H} = \{1, 2, \ldots, n\}$.*

That the set of operators $\{M_e\}$ in Eqs. (14) actually describes a measurement follows by just checking the normalization (3). Moreover, the set of operators satisfies condition (6), as well as condition (8), whence they describe a repeatable measurement. On the other hand, the measurement is not orthogonal, since the corresponding positive operator valued measurement (POVM) is given by

$$P_l = p_l|0\rangle\langle 0| + \sum_{j=0}^{\infty} |nj + l\rangle\langle nj + l|, \qquad 1 \leq l \leq n. \tag{15}$$

We emphasize that the same POVM also describes a nonrepeatable measurement, such as that corresponding to the set of contractions

$$N_l = \sqrt{p_l}|0\rangle\langle 0| + \sum_{j=0}^{\infty} |nj + l\rangle\langle nj + l|, \qquad 1 \leq l \leq n. \tag{16}$$

This fact evidences that repeatability is a feature which is obviously related to the state reduction of the measurement, not to the POVM, e.g., one can have an orthogonal POVM for a nonrepeatable measurement.

At this point, the question is how to characterize a generic nonorthogonal repeatable measurement, namely, which is the general form of the contractions $\{M_e\}$ that satisfies Eq. (6) or (8). The necessary conditions (9) now come at hand: If we exclude the case of orthogonal measurements, then there must exist at least one $M_i$ such that one has the strict inclusion $\text{Rng}(M_i) \subset \text{Supp}(M_i)$. We can now decompose the subspace $\text{Supp}(M_i)$ in orthogonal components as follows:

$$\mathsf{Supp}(M_i) = \text{Rng}(M_i) \oplus C(M_i), \tag{17}$$

where $C(M_i)$ is the orthogonal complement of $\text{Rng}(M_i)$ in $\text{Supp}(M_i)$. The operator $M_i$ on its support can then be written as

$$M_i = V_i + W_i, \tag{18}$$

with $\text{Supp}(V_i) = \text{Rng}(M_i)$ and $\text{Supp}(W_i) = C(M_i)$. The normalization of the POVM implies

$$\sum_e (V_e^\dagger V_e + W_e^\dagger W_e + V_e^\dagger W_e + W_e^\dagger V_e) = I. \tag{19}$$

Since they represent off-diagonal operators, the cross terms must be null. More precisely, one must have $W_e^\dagger V_e = 0$ for each term separately, since due to orthogonality of supports for different $e$ these terms are all linearly independent, and similarly $V_e^\dagger W_e = 0$ by orthogonality of ranges. These facts along with Eq. (6)—which states that $M_i$ is isometric on its range—implies that $V_i$ is a partial isometry,

$$V_i^\dagger V_i = P_i \equiv P_{\text{Rng}(M_i)}, \tag{20}$$

and we can rewrite the normalization condition (3) as

$$\sum_e P_e + \sum_f W_f^\dagger W_f = I. \tag{21}$$

The only conditions that the operators $W_e$ must obey are then

$$\begin{aligned} &\mathsf{S}\text{upp}(W_e) = C(M_e), \\ &\text{Rng}(W_e) \subseteq \text{Rng}(M_e) \Rightarrow W_f^\dagger W_e = 0, \qquad e \neq f, \\ &\sum_e W_e^\dagger W_e = P_{\mathsf{K}}, \end{aligned} \tag{22}$$

where $P_{\mathsf{K}}$ is the projection on the intersection space $\mathsf{K} = [\oplus_e \text{Rng}(M_e)]^\perp$. For some events $f$, the operator $W_f$ could be null, namely, $\text{Supp}(M_f) \equiv \text{Rng}(M_f)$: When this holds for all events $f \in \mathcal{H}$, the described measurement is just the conventional orthogonal one. Summarizing, for a nonorthogonal repeatable measurement, the contractions $M_e$ have supports that intersect, at least for a couple of events $e$, but their ranges fall outside the intersection, as represented in Fig. 1.

They act as an isometry on their ranges, while on the intersection space $\mathsf{K}$ it is the sum of $W_e^\dagger W_e$ that acts as the identity. Notice that each operator $W_e$ needs not be singularly proportional to a partial isometry, as in the example given before. In fact, consider the binary measurement described by the contractions

$$M_1 = \sqrt{p_1}|2\rangle\langle 0| + \sqrt{p_2}|4\rangle\langle 1| + \sum_{n=1}^{\infty} |2(n + 2)\rangle\langle 2n|,$$

$$M_2 = \sqrt{1 - p_1}|3\rangle\langle 0| + \sqrt{1 - p_2}|5\rangle$$

$$\times \langle 1| + \sum_{n=1}^{\infty} |2(n + 2) + 1\rangle\langle 2n + 1|. \tag{23}$$
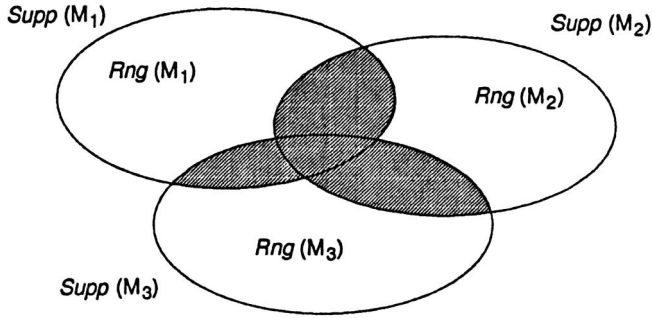
The corresponding POVM is given by

FIG. 1. Illustration of the relations between supports and ranges of the contractions of a generic repeatable nonorthogonal measurement with three outcomes.

$$P_1 = p_1|0\rangle\langle 0| + p_2|1\rangle\langle 1| + \sum_{n=1}^{\infty} |2n\rangle\langle 2n|,$$

$$P_2 = (1 - p_1)|0\rangle\langle 0| + (1 - p_2)|1\rangle \qquad (24)$$

$$\times \langle 1| + \sum_{n=1}^{\infty} |2n+1\rangle\langle 2n+1|.$$

In this case

$$W_1^\dagger W_1 = p_1|0\rangle\langle 0| + p_2|1\rangle\langle 1|,$$
$$W_2^\dagger W_2 = (1 - p_1)|0\rangle\langle 0| + (1 - p_2)|1\rangle\langle 1|, \qquad (25)$$

while, clearly,

$$W_1^\dagger W_1 + W_2^\dagger W_2 = |0\rangle\langle 0| + |1\rangle\langle 1| = P_\mathsf{K}. \qquad (26)$$

We are now in position to state the general form of a POVM $\{P_e\}$ admitting a repeatable measurement. One must have

$$P_e = Z_e + T_e, \qquad e \in \mathcal{H}, \qquad Z_e T_f = T_f Z_e = 0,$$

$$\forall\, e, f \in \mathcal{H}, \qquad T_e \geq 0, \qquad \sum_{e \in \mathcal{H}} T_e = Z_\omega, \qquad (27)$$

$$Z_e Z_f = Z_e \delta_{ef}, \qquad \forall\, e, f \in \mathcal{H} \cup \{\omega\},$$

with the normalization $\sum_{e \in \mathcal{H}} P_e \equiv Z_\omega + \sum_{e \in \mathcal{H}} Z_i = I$. The orthogonal case corresponds to $T_e = 0$, $\forall\, e \in \mathcal{H}$.

Let us now see how a memory of the number of performed repetitions is associated to a nonorthogonal repeatable measurement. This is a consequence of a theorem by Wold and von Neumann [9,10] which states that every isometry can be written as a direct sum of unilateral shift operators and possibly a unitary (a unilateral shift $S$ can always be written in the form $S = \sum_{j=1}^{\infty} |j + k\rangle\langle j|$, $k \geq 1$, for a suitable orthonormal basis $\{|j\rangle\}$). The operators $V_e$ in Eq. (18) can be further separated in the direct sum $V_e + U_e + S_e$ of a unitary $U_e$ and a pure isometry $S_e$, and we have

$$M_e + V_e + W_e = U_e + S_e + W_e. \qquad (28)$$

Let us now consider an initial state $|\psi\rangle$ with nonvanishing

component in the support of $M_e$, and suppose that the outcome $e$ occurred. Since $V_e^\dagger W_e = 0$, one can equivalently write $S_e^\dagger W_e = 0$ and $U_e^\dagger W_e = 0$. The latter identity implies that the range of $W_e$ is orthogonal to $\mathrm{Supp}(U_e^\dagger) \equiv \mathrm{Supp}(U_e)$, and thus the conditional state $|\psi_e\rangle = (M_e|\psi\rangle)/(\||M_e|\psi\rangle\|)$ cannot be in the support of $U_e$, namely, it must belong to the support of $S_e$. Therefore, for the successive measurements, we will effectively have $M_e \equiv S_e$, and successive applications will shift the observable $\{|j\rangle\langle j|\}$ to $\{|j - k\rangle\langle j - k|\}$, where $\{|j\rangle\}$ is the orthonormal shifted basis for any chosen unilateral shift component of $S_e$. Notice that the index $j$ can be checked without affecting the repeatability of the outcome $e$.

In summary, we have shown that there exist nonorthogonal perfectly repeatable measurements, and only for finite dimensions repeatability is equivalent to orthogonality. On the contrary, for infinite dimension there exist nonorthogonal repeatable measurements, of which we have given the most general form, based on necessary and sufficient conditions, and providing some explicit examples. Finally, we have shown how the measured system undergoing such a measurement must retain some memory of the number of times that the measurement was performed.

*URL: http://www.qubit.it
[1] P. Mittelstaedt, *The Interpretation of Quantum Mechanics and the Measurement Process* (Cambridge University Press, Cambridge, England, 1998).
[2] J. Von Neumann, *Mathematical Principles of Quantum Mechanics* (Princeton University Press, Princeton, NJ, 1955).
[3] G. Lüders, Ann. Phys. (Berlin) **8**, 322 (1951).
[4] E. B. Davies and J. T. Lewis, Commun. Math. Phys. **17**, 239 (1970).
[5] M. Ozawa, J. Math. Phys. (N.Y.) **25**, 79 (1984).
[6] M. Ozawa, Publ. Res. Inst. Math. Sci. **21**, 279 (1985).
[7] The fact that orthogonal projection follows from further assumptions in addition to repeatability was emphasized by Ozawa in many papers; see, for example, Ref. [8].
[8] M. Ozawa, Phys. Rev. A **62**, 062101 (2000).
[9] G. J. Murphy, *C\*-Algebras and Operator Theory* (Academic, London, 1990).
[10] P. R. Halmos, *A Hilbert Space Problem Book* (American Book, New York, 1967).