Information on Quantum Devices

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Entanglement as a resource to retrieve information on devices

- Complete experimental characterization of quantum devices.
- Discrimination between Quantum Operations.

References

- G. M. D'Ariano, and P. Lo Presti, *Quantum tomography for measuring experimentally the matrix elements of an arbitrary quantum operation*, Phys. Rev. Lett. **86** 4195 (2001)
- G. M. D'Ariano, and P. Lo Presti, M. G. A. Paris, *Using entanglement improves precision of quantum measurements* Phys. Rev. Lett. **87** 270404 (2001); *Improved discrimination of unitary transformations by entangled probes*, J. Opt. B **4** S273 (2002)
- G. M. D'Ariano and P. Lo Presti, *Imprinting a complete information about a quantum channel on its output state*, quant-ph/0211133

Why entanglement?

- Answer: **the entangled state works effectively as all possible input states in "quantum parallel".**

Entangled states

- Entangled states $|\Psi\rangle\rangle \in \mathsf{H} \otimes \mathsf{H}$

$$|\Psi\rangle\rangle = \sum_{nm} \Psi_{nm} |n\rangle \otimes |m\rangle.$$

- Matrix notation (for fixed reference basis in the two Hilbert spaces):

$$A \otimes B|C\rangle\rangle = |AC\, B^{\tau}\rangle\rangle,$$

$$|A\rangle\rangle \doteq \sum_{nm} A_{nm} |n\rangle \otimes |m\rangle \equiv A \otimes I|I\rangle\rangle \equiv I \otimes A^{\tau}|I\rangle\rangle \ ,$$

$$|I\rangle\rangle = \sum_{n} |n\rangle \otimes |n\rangle.$$

- Isomorphism $\mathsf{HS}(\mathsf{H}) \simeq \mathsf{H} \otimes \mathsf{H}$ between the Hilbert space $\mathsf{HS}(\mathsf{H})$ of Hilbert-Schmidt operators on H and $\mathsf{H} \otimes \mathsf{H}$

$$\langle\langle A|B\rangle\rangle \equiv \mathrm{Tr}[A^{\dagger}B].$$

- Measure of the entanglement for pure states: von Neumann entropy $S(\rho) = -\,\mathrm{Tr}[\rho \ln \rho]$ of the local state

$$\rho = \mathrm{Tr}_2[|\Psi\rangle\rangle\langle\langle\Psi|] \equiv \Psi\Psi^{\dagger}.$$

Quantum operations

- The most general state (conditioned) evolution in quantum mechanics:

  the "quantum operation" (Kraus)

$$\rho \to \frac{\mathcal{E}(\rho)}{\mathrm{Tr}[\mathcal{E}(\rho)].}$$

- The **quantum operation** $\mathcal{E}$ is a map on traceclass operators that is
  1. linear
  2. trace-decreasing
  3. completely positive
- The normalization $\mathrm{Tr}[\mathcal{E}(\rho)] \leq 1$ is the probability that the transformation occurs.

Quantum operations: examples

1. Unitary transformations:

$$\mathcal{E}(\rho) = U\rho U^{\dagger}.$$

2. Pure operations:

$$\mathcal{E}(\rho) = A\rho A^{\dagger} \ ,$$

   A contraction, i. e. $\|A\| \leq 1$.

3. Mixing transformations:

$$\mathcal{E}(\rho) = \sum_n K_n \rho K_n^{\dagger}.$$

4. Deterministic transformations (channels):

$$\mathrm{Tr}[\mathcal{E}(\rho)] = \mathrm{Tr}[\rho] \Rightarrow \sum_n K_n^{\dagger} K_n = I \ .$$

Problems

- Suppose now that we have a quantum machine that performs an unknown quantum operation $\mathcal{E}$.

- Problems:

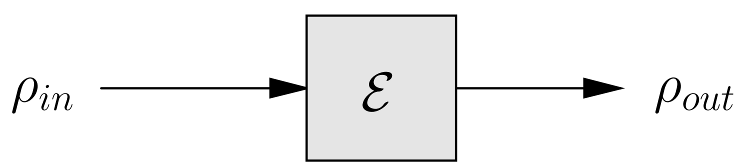
  1. How to determine $\mathcal{E}$ experimentally?
  2. If we have a known set of operations $\mathcal{E}_1, \mathcal{E}_2, \ldots$, how to discriminate among them?

  Case 1 is the complete quantum characterization of a device.

  Case 2 corresponds to event detecting, measurement of parameters, etc.

Howto characterize the operation of a device

- Any linear device (e.g. optical lens, amplifier) can be completely described by a **transfer matrix** which gives the vector output by matrix-multiplying the vector input.
- $\boxed{\text{Problem}}$: **how to reconstruct the full transfer matrix of a device?**
- Answer (brute force): **by scanning a *basis* of possible inputs**, and measuring the corresponding outputs.



- In quantum mechanics the inputs and outputs are density operators, and the role of the transfer matrix is played by the quantum operation of the device (which is linear a part from a normalization).
- We need to run a complete orthogonal basis of quantum states $|n\rangle$ $(n = 0, 1, 2, \ldots)$, along with their linear combinations $\frac{1}{\sqrt{2}}(|n'\rangle + i^k|n''\rangle)$, with $k = 0, 1, 2, 3$ and i denoting the imaginary unit.
- However, **the availability of a basis of states in the lab is a very hard technological problem**.

Complete positivity: relevant theorems

- One-to-one correspondence $\mathcal{E} \leftrightarrow R_{\mathcal{E}}$ between quantum operations on $\mathsf{T}(\mathsf{H})$ and positive operators $R_{\mathcal{E}}$ on $\mathsf{H} \otimes \mathsf{H}$:

$$R_{\mathcal{E}} = \mathcal{E} \otimes \mathcal{I}_{\mathsf{H}}(|I\rangle\!\rangle\langle\!\langle I|) \ ,$$
$$\mathcal{E}(\rho) = \mathrm{Tr}_2[(I \otimes \rho^{\tau})R_{\mathcal{E}}] \ ,$$

where

$$|I\rangle\!\rangle = \sum_n |n\rangle \otimes |n\rangle, \ \{|n\rangle\} \text{ orthonormal basis}$$

- The most general form for $\mathcal{E}$ is (Kraus)

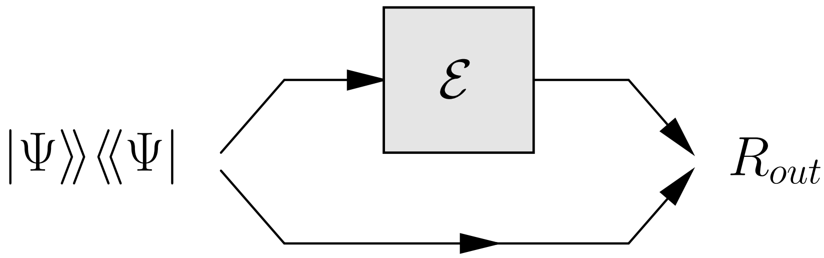
$$\mathcal{E}(\rho) = \sum_n K_n \rho K_n^{\dagger} \ ,$$

where the operators K_n satisfy the bound

$$\sum_n K_n^{\dagger} K_n \leq I \ .$$

The entangled input

- $\boxed{\text{Quantum parallelism of entanglement}}$: **a single entangled input state $|\Psi\rangle\rangle$ is equivalent to scanning all states in parallel**.



- We need to put the entangled state at the input of the device with two identical quantum systems prepared in an entangled state $|\Psi\rangle\rangle$, and only one of the two systems undergoing the quantum operation $\mathcal{E}$, whereas the other is left untouched.

- In tensor-product notation this setup is expressed as the quantum operation

$$R_{out} = \mathcal{E} \otimes \mathcal{I}(|\Psi\rangle\rangle\langle\langle\Psi|).$$

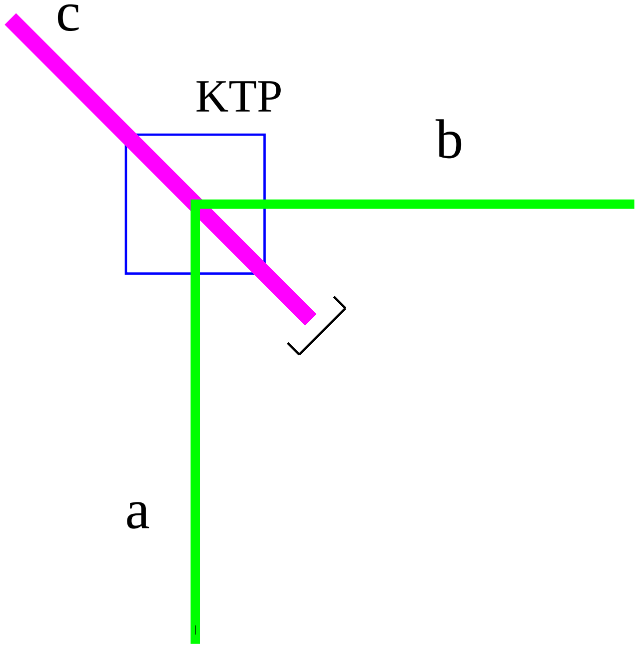
where the entangled state $|\Psi\rangle\rangle$ is given by

$$|\Psi\rangle\rangle = \sum_{nm} \Psi_{nm}|n\rangle \otimes |m\rangle.$$

- For fixed *faithful* state $|\Psi\rangle\rangle$ (Ψ full-rank) the output state $R_{out} \equiv R_{\mathcal{E}}(\Psi)$ is in one-to-one correspondence with the quantum operation of the device $\mathcal{E}$.

Availability of the entangled input

- Full-rank entangled states can be easily generated in Quantum Optics from **parametric downconversion of vacuum**



- Hamiltonian $\boxed{H \propto c a^\dagger b^\dagger + h.c.}$ where $\omega_c = \omega_a + \omega_b$.

- From input vacuum in a and b and classical pump c produces the "twin-beam"

$$\boxed{|\Psi\rangle\!\rangle = (1 - |\xi|^2)^{\frac{1}{2}} \sum_{n=0}^{\infty} \xi^n |n\rangle \otimes |n\rangle}$$

- Faithful entangled states of *qubits* can be generated by means of networks of controlled-NOT gates.

Quantum tomography

- Howto determine the output state?

  Answer: using quantum tomography.

- Quantum tomography is a method to estimate the ensemble average $\langle H \rangle$ of any arbitrary operator H on H by using only measurement outcomes of a *quorum* of observables $\{O_l\}$.

- The density matrix corresponds to estimating the ensemble averages of $|i\rangle\langle j|$.

- This means that any operator H can be expanded as

$$H = \sum_l \langle Q_l, H \rangle O_l,$$

  for suitable scalar product $\langle , \rangle$ and dual set $\{Q_l\}$.

- Hence, the tomographic estimation of the ensemble average $\langle H \rangle$ is obtained as double averaging over both the ensemble and the quorum.

- Very powerful experimental method. General approach for unbiasing the instrumental noise. Improvements based on adaptive techniques, maximum-likelihood strategies, etc.

- For multipartite quantum systems, simply a quorum is the tensor product of single-system quorums: this means that, in our case, we just need to make two local quorum measurements jointly on the two systems.

Homodyne tomography

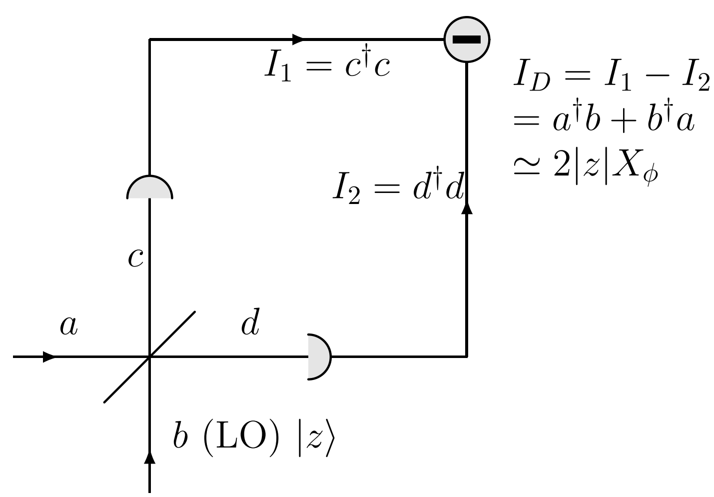
- In quantum optics a *quorum* for each mode of the field is given by the set of *quadratures*

$$X_\phi = \frac{1}{2}\left(a^\dagger e^{i\phi} + a e^{-i\phi}\right) .$$

- One has

$$\begin{aligned} \langle H \rangle &= \int_0^\pi \frac{\mathrm{d}\,\phi}{\pi} \langle E_H(X_\phi;\phi)\rangle \ , \\ E_H(x;\phi) &= \frac{1}{4}\int_{-\infty}^{+\infty} \mathrm{d}\,k\,|k| \mathrm{Tr}[H e^{ikX_\phi}] e^{-ikx} \ , \end{aligned}$$

- Balanced homodyne detection



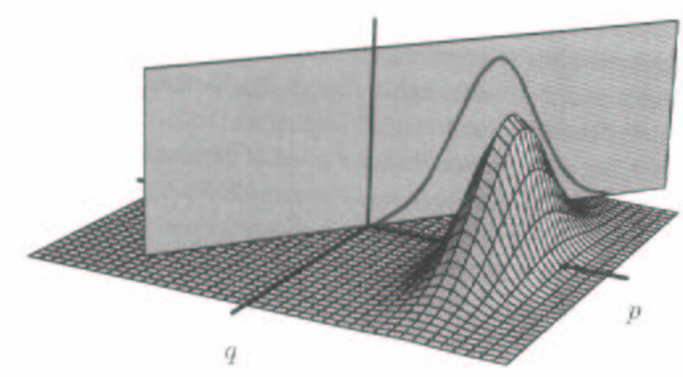
$$c = \frac{1}{\sqrt{2}}(a+b) \ , \quad d = \frac{1}{\sqrt{2}}(a-b) \ .$$

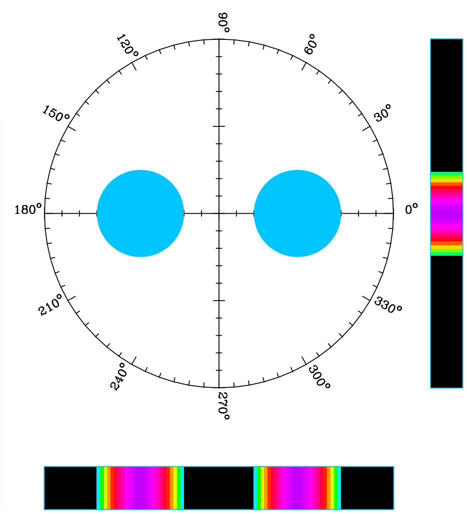
- In the *strong LO limit* ($z \to \infty$) a balanced homodyne detector measures the quadrature X_ϕ of the field at any desired phase ϕ with respect to the local oscillator (LO).

Homodyne tomography

$$\begin{aligned}\langle H \rangle &= \int_0^{\pi} \frac{\mathrm{d}\,\phi}{\pi} \langle E_H(X_\phi;\phi)\rangle \ ,\\ E_H(x;\phi) &= \frac{1}{4}\int_{-\infty}^{+\infty} \mathrm{d}\,k\,|k| \mathrm{Tr}[H e^{ikX_\phi}] e^{-ikx} \ ,\end{aligned}$$

- Analogy with the Radon transform for *imaging*





- A **tomography** of a two dimensional image $W(\alpha,\bar{\alpha})$ is a collection of one dimensional projections $p(x;\phi)$ at different values of the observation angle ϕ.

$$W(\alpha,\overline{\alpha}) = \int_{-\infty}^{+\infty} \frac{dr|r|}{4} \int_0^{\pi} \frac{d\phi}{\pi} \int_{-\infty}^{+\infty} dx\, p(x;\phi) \exp\left[ir(x-\alpha_\phi)\right] \ .$$

Pauli Tomography

Pauli matrices with identity $I\ ,\sigma_x\ ,\sigma_y\ ,\sigma_z$: orthonormal basis for the qubit operator space:

$$H = \frac{1}{2}\{\sigma \cdot \mathrm{Tr}[\sigma H] + I\,\mathrm{Tr}[H]\}\ .$$

- Tomographic estimation:

$$\langle H \rangle = \frac{1}{3} \sum_{\alpha=x,y,z} \langle E_H(\sigma_\alpha;\alpha)\rangle\ ,$$

$$E_H(\sigma_\alpha;\alpha) = \frac{3}{2}\mathrm{Tr}[H\sigma_\alpha]\sigma_\alpha + \frac{1}{2}\mathrm{Tr}[H]$$

Pauli Tomography

- Qubit realized by polarization of single photon states.

$$\sigma_z = h^\dagger h - v^\dagger v,$$
$$|\uparrow\rangle \equiv |1\rangle_h |0\rangle_v, \; |\downarrow\rangle \equiv |0\rangle_h |1\rangle_v,$$

$$\sigma_y = e^{i\frac{\pi}{4}\sigma_x}\sigma_z e^{-i\frac{\pi}{4}\sigma_x} \; ,$$
$$e^{-i\frac{\pi}{4}\sigma_x}|1\rangle_h|0\rangle_v = \frac{1}{\sqrt{2}}[|1\rangle_h|0\rangle_v - i|0\rangle_h|1\rangle_v] \equiv |1\rangle_l|0\rangle_r \; ,$$

$$\sigma_x = e^{-i\frac{\pi}{4}\sigma_y}\sigma_z e^{i\frac{\pi}{4}\sigma_y} \; ,$$
$$e^{i\frac{\pi}{4}\sigma_y}|1\rangle_h|0\rangle_v = \frac{1}{\sqrt{2}}[|1\rangle_h|0\rangle_v - |0\rangle_h|1\rangle_v] \equiv |1\rangle_{\nearrow}|0\rangle_{\nwarrow} \; .$$

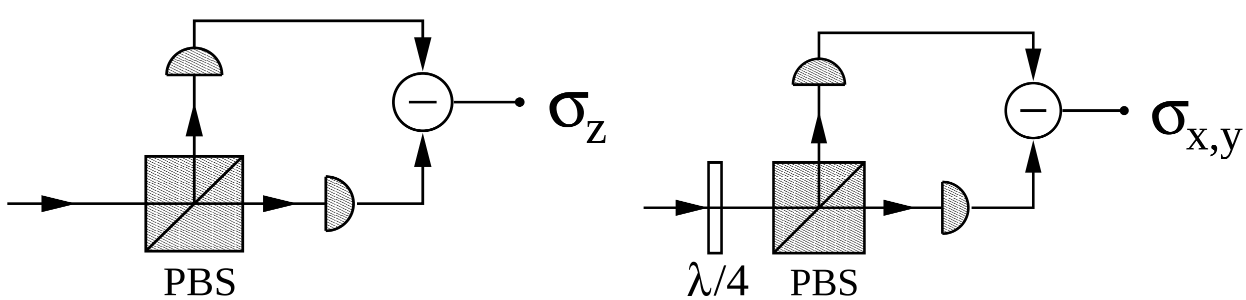


Figure 1: Pauli-matrix detectors for photon-polarization qubits.

Some experimental results

- First measurement of the joint photon-number probability distribution for a two-mode quantum state created by a nondegenerate optical parametric amplifier.

  [M. Vasilyev, S.-K. Choi, P. Kumar, and G. M. D'Ariano, *Tomographic Measurement of Joint Photon Statistics of the Twin-Beam Quantum State*, Phys. Rev. Lett. **84** 2354 (2000)]

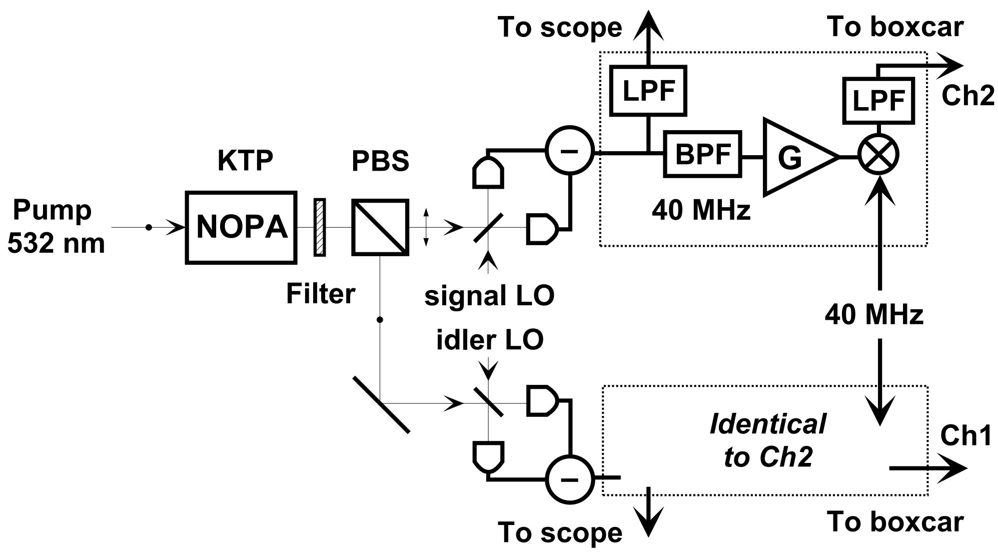


Figure 2: A schematic of the experimental setup. NOPA, non-degenerate optical parametric amplifier; LOs, local oscillators; PBS, polarizing beam splitter; LPFs, low-pass filters; BPF, band-pass filter; G, electronic amplifier. Electronics in the two channels are identical. The measured distributions exhibit up to 1.9 dB of quantum correlation between the signal and idler photon numbers, whereas the marginal distributions are thermal as expected for parametric fluorescence.

Some experimental results

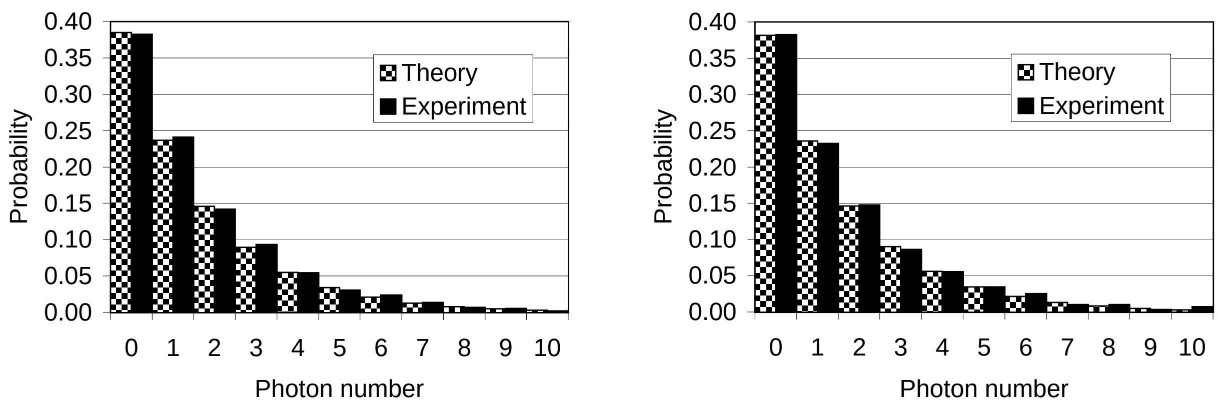


Figure 3: Marginal distributions for the signal and idler beams. Theoretical distributions for the same mean photon numbers are also shown.

Some experimental results

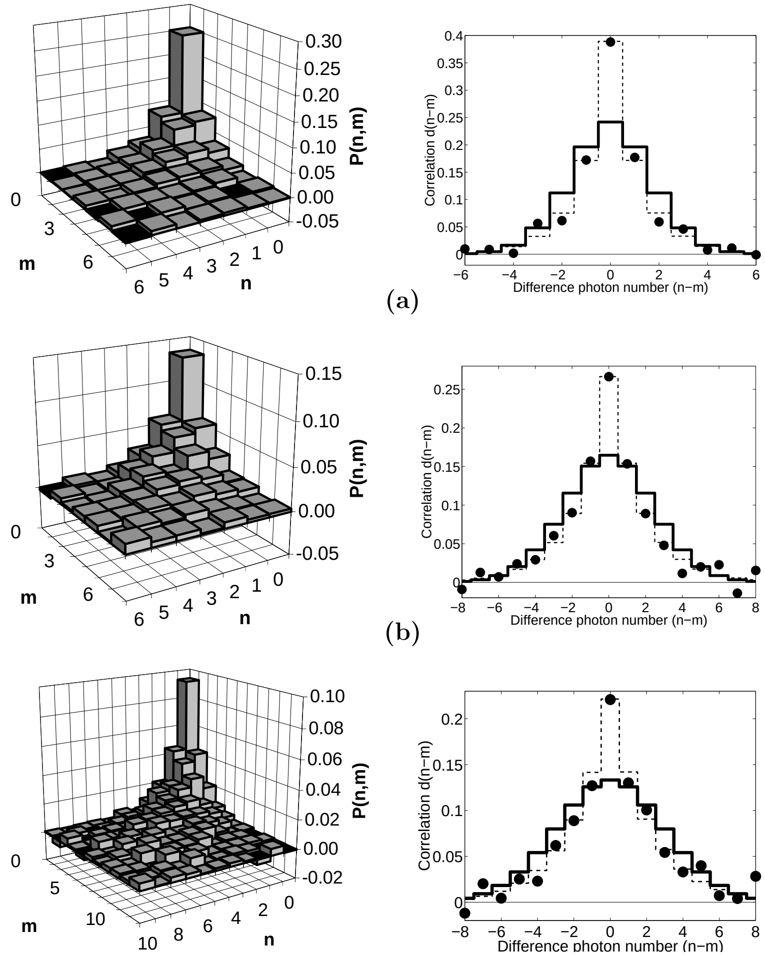
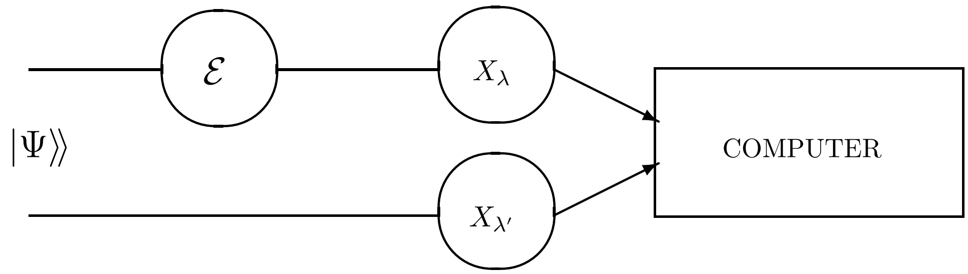


Figure 4: Left: Measured joint photon-number probability distributions for the twin-beam state. Right: Difference photon number distributions corresponding to the left graphs (filled circles, experimental data; solid lines, theoretical predictions; dashed lines, difference photon-number distributions for two independent coherent states with the same total mean number of photons and $\overline{n} = \overline{m}$. (a) 400000 samples, $\overline{n} = \overline{m} = 1.5$, $N = 10$; (b) 240000 samples, $\overline{n} = 3.2$, $\overline{m} = 3.0$, $N = 18$; (c) 640000 samples, $\overline{n} = 4.7$, $\overline{m} = 4.6$, $N = 16$.

Tomography of quantum operations



- **General method:** Two identical quantum systems are prepared in an entangled state $|\Psi\rangle\rangle$. One of the two systems undergoes the quantum operation $\mathcal{E}$, whereas the other is left untouched. At the output one makes a quantum tomographic estimation, photocurrent by measuring jointly two random observables from a quorum $\{X_\lambda\}$.
- The output state is the joint density matrix

$$|\Psi\rangle\rangle\langle\langle\Psi| \to R(\Psi) \equiv \mathcal{E} \otimes \mathcal{I}(|\Psi\rangle\rangle\langle\langle\Psi|).$$

- The quantum operation $\mathcal{E}$ is in correspondence with $R_{\mathcal{E}} \equiv R(\Psi)$ for $\Psi = I$, and for invertible Ψ the two matrices $R(I)$ and $R(\Psi)$ are connected as follows

$$R(I) = (I \otimes \Psi^{-1^\tau})R(\Psi)(I \otimes \Psi^{-1*}) \ .$$

Hence, the quantum operation (four-index) matrix $R_{\mathcal{E}}$ can be obtained by estimating via quantum tomography the following output ensemble averages

$$\langle\langle i, j|R(I)|l, k\rangle\rangle = \left\langle \ |l\rangle\langle i| \otimes \Psi^{-1*}|k\rangle\langle j|\Psi^{-1^\tau} \ \right\rangle \ .$$

Tomography of quantum operations

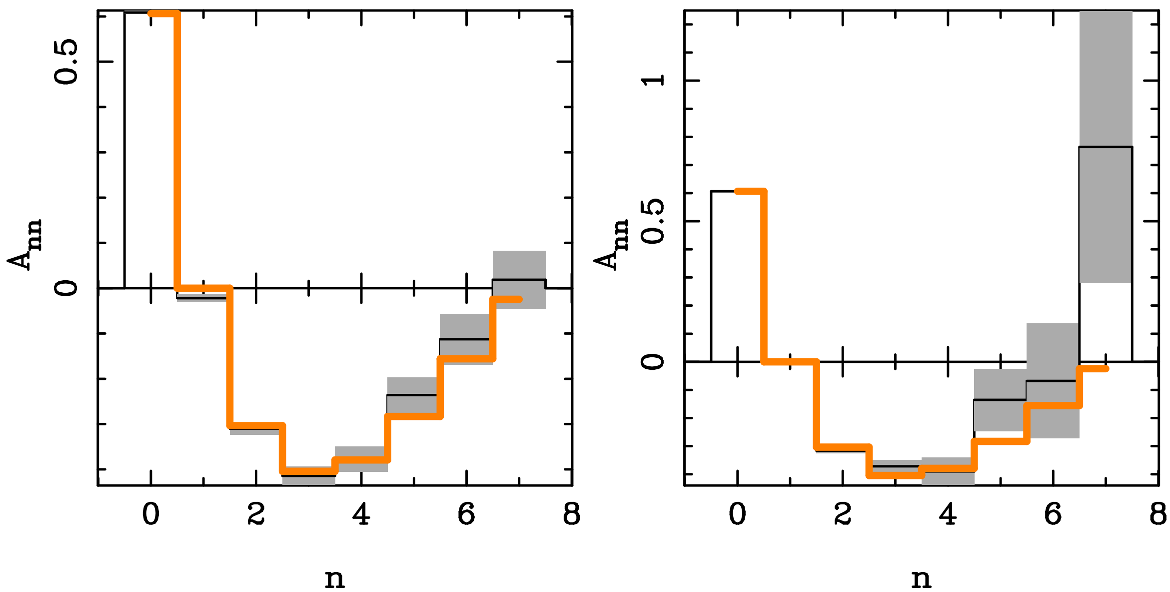


Figure 5: Homodyne tomography of the quantum operation A corresponding to the unitary displacement of one mode of the radiation field. Diagonal elements A_{nn} (shown by thin solid line on an extended abscissa range,) with their respective error bars in gray shade, compared to the theoretical probability (thick solid line). Similar results are obtained for all upper and lower diagonals of the quantum operation matrix A. The reconstruction has been achieved using an entangled state $|\psi\rangle\rangle$ at the input corresponding to parametric downconversion of vacuum with mean thermal photon $\bar{n}$ and quantum efficiency at homodyne detectors η. Top: $z = 1$, $\bar{n} = 5$, $\eta = 0.9$, and 150 blocks of 10^4 data have been used. Bottom: $z = 1$, $\bar{n} = 3$, $\eta = 0.7$, and 300 blocks of $2 \cdot 10^5$ data have been used. The bottom plot corresponds to the same parameters of the experiment in Ref. M. Vasilyev, S.-K. Choi, P. Kumar, and G. M. D'Ariano, Phys. Rev. Lett. **84** 2354 (2000).

- The method exploits the *quantum parallelism* of entanglement, with **a single entangled state playing the role of a varying input state**, thus overcoming the practically unsolvable problem of availability of all possible input states for the tomographic analysis of the quantum operation.

Tomography of a single qubit quantum device

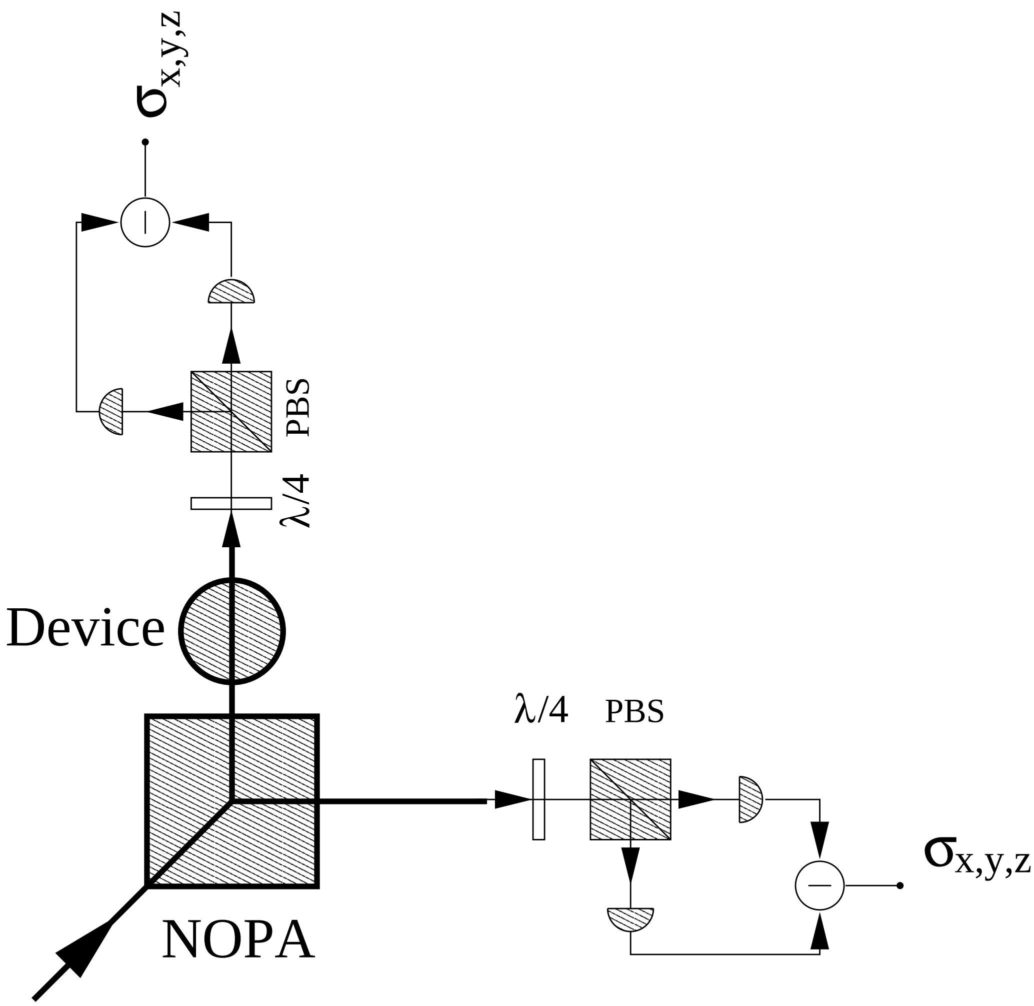
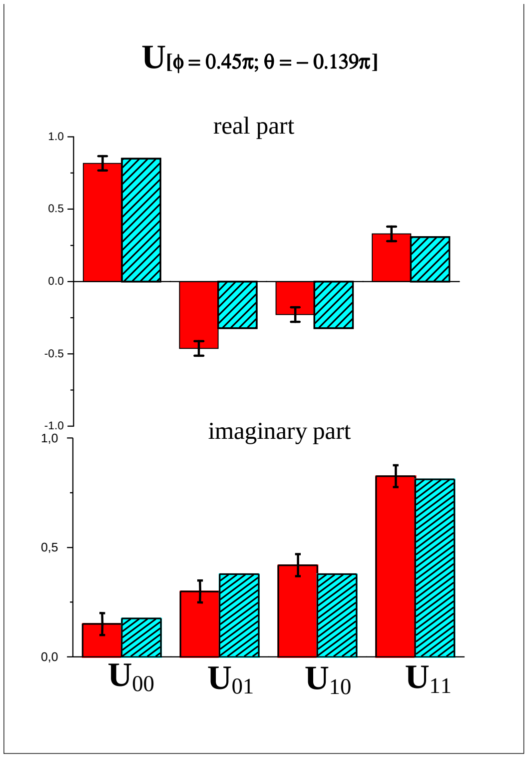
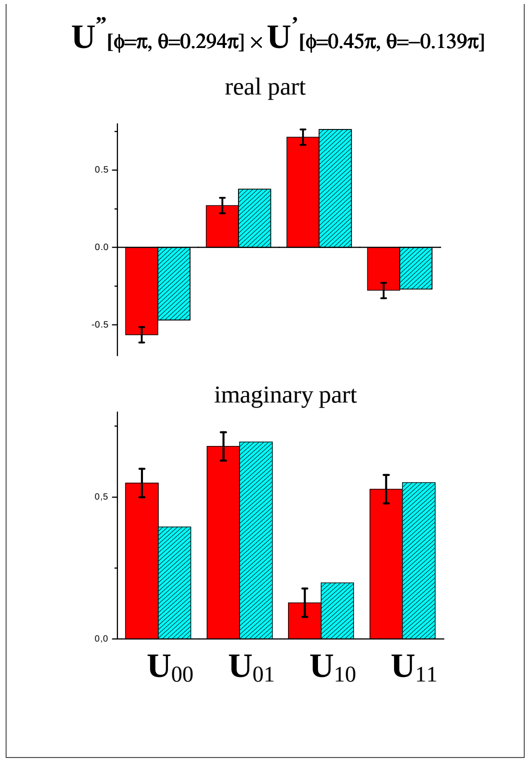


Figure 6: Experiment in progress in Roma La Sapienza, F. De Martini lab.

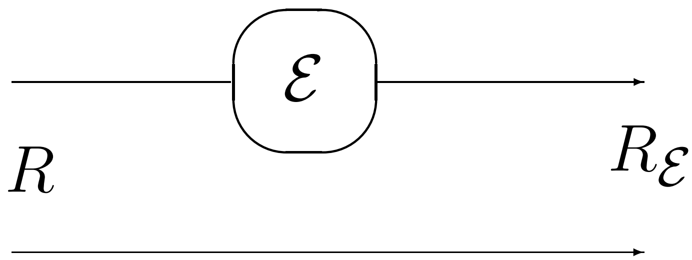
Tomography of a single qubit quantum device





Faithful states

- Is it possible to characterize a quantum operation using mixed entangled states, or even separable ones?
- Answer: yes, as long as the state is *faithful*.
- We call a bipartite state faithful when acting with a channel on one of the two quantum systems, the output state carries a complete information about the channel.

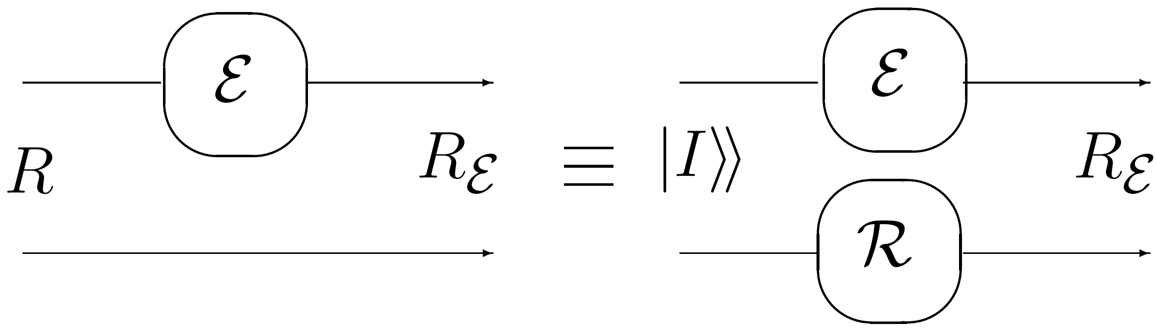


$$R_{\mathcal{E}} \doteq \mathcal{E} \otimes \mathcal{I}(R).$$

Namely: the input state R is called *faithful* when the correspondence between the output state $R_{\mathcal{E}} \doteq \mathcal{E} \otimes \mathcal{I}(R)$ and the quantum channel $\mathcal{E}$ is one-to-one.

- The concept of faithfulness can also be extended to sets of states, when the output states patched together carry a complete imprinting of the channel.

Faithful states



$$R = \sum_l |A_l\rangle\rangle\langle\langle A_l| = \mathcal{I} \otimes \mathcal{R}(|I\rangle\rangle\langle\langle I|),\ \mathcal{R}(\rho) = \sum_l A_l^\tau \rho A_l^*.$$

- A state R is faithful when the map $\mathcal{R}$ is invertible, in order to guarantee the one-to-one correspondence between $R_{\mathcal{E}}$ and $\mathcal{E}$.

- The information about the channel $\mathcal{E}$ can be extracted from $R_{\mathcal{E}}$ as follows

$$\mathcal{E}(\rho) = \mathrm{Tr}_2[(I \otimes \rho^\tau)\mathcal{I} \otimes \mathcal{R}^{-1}(R_{\mathcal{E}})].$$

- A pure state $R \equiv |A\rangle\rangle\langle\langle A|$ is faithful iff it has maximal Schmidt's number.

- The set of faithful states R is *dense* within the set of all bipartite states.

- However, the knowledge of the map $\mathcal{E}$ from a measured $R_{\mathcal{E}}$ will be affected by increasingly large statistical errors for $\mathcal{R}$ approaching a non-invertible map.

- It follows that there are faithful states among mixed separable states.

Examples of faithful states

- Werner's states:

$$R_f = \frac{1}{d(d^2-1)}[(d-f)+(df-1)E],$$

E swap operator, $d = \dim(\mathsf{H})$, $(-1 \le f \le 1)$

- faithful for all $f \neq \frac{1}{d}$, separable for $f \ge 0$.

- Isotropic states for dimension d

$$R_f = \tfrac{f}{d}|I\rangle\!\rangle\langle\!\langle I| + \tfrac{1-f}{d^2-1}(I - \tfrac{1}{d}|I\rangle\!\rangle\langle\!\langle I|),$$

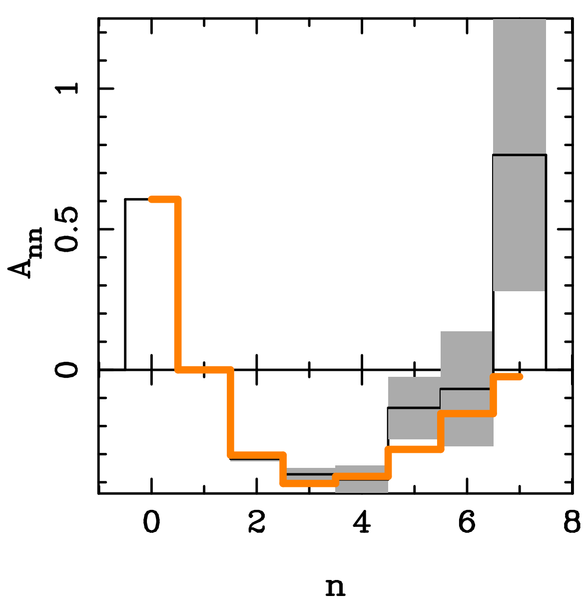
- faithful for $f \neq \frac{1}{d^2}$, separable for $f \le \frac{1}{d}$.

Faithful states for "continuous variables"

- The inverse map $\mathcal{R}^{-1}$ is unbounded.
- As a result we will recover the channel $\mathcal{E}$ from the measured $R_{\mathcal{E}}$ with unbounded amplification of statistical errors, (depending on the chosen representation).
- Example: twin beam from parametric down-conversion of vacuum

$$|\Psi\rangle\!\rangle = \Psi \otimes I|I\rangle\!\rangle, \quad \Psi = (1-|\xi|^2)^{\frac{1}{2}}\xi^{a^\dagger a}, \qquad |\xi| < 1.$$

- The state is faithful, but the operator Ψ^{-1} is unbounded, whence the inverse map $\mathcal{R}^{-1}$ is also unbounded.
- For example, in a photon number representation $\mathsf{B} = \{|n\rangle\langle m|\}$, the effect will be an amplification of errors for increasing numbers n, m of photons.



Faithful states for "continuous variables"

- Consider now the quantum channel describing the *Gaussian displacement noise*

$$\mathcal{N}_\nu(\rho) = \int_{\mathbb{C}} \frac{\mathrm{d}\,\alpha}{\pi\nu} e^{-\frac{|\alpha|^2}{\nu}} D(\alpha)\rho D^\dagger(\alpha),$$

- analogous of the depolarizing channel for infinite dimension.

- Multiplication rule

$$\mathcal{N}_\nu \mathcal{N}_\mu = \mathcal{N}_{\nu+\mu},$$

whence the inverse map is formally given by

$$\mathcal{N}_\nu^{-1} \equiv \mathcal{N}_{-\nu}.$$

- As a faithful state consider now the mixed state given by the twin-beam, with one beam spoiled by the Gaussian noise, namely

$$R = \mathcal{I} \otimes \mathcal{N}_\nu(|\Psi\rangle\rangle\langle\langle\Psi|).$$

One has

$$R = \frac{1}{\nu}(\Psi \otimes I) \exp\left[-\frac{(a-b^\dagger)(a^\dagger - b)}{\nu}\right](\Psi^\dagger \otimes I),$$

and its partial transposed

$$R^{\tau_2} = (\nu+1)^{-1}(\Psi \otimes I)\left(\frac{\nu-1}{\nu+1}\right)^{\frac{1}{2}(a-b)^\dagger(a-b)}(\Psi^\dagger \otimes I),$$

- Since our state is Gaussian, the PPT criterion guarantees separability,[1] and for $\nu > 1$ our state is separable, still it is *formally* faithful, since the operator Ψ and the map N_ν are both invertible.

[1] R. Simon, Phys. Rev. Lett. **84**, 2726 (2000)

Faithful states for "continuous variables"

- Unboundedness of the inverse map can wash out completely the information on the channel in some particular chosen representation.
- Example: (overcomplete) representation $\mathsf{B} = \{|\alpha\rangle\langle\beta|\}$, with $|\alpha\rangle$ and $|\beta\rangle$ coherent states.

- From the identity

$$\mathcal{N}_\nu(|\alpha\rangle\langle\alpha|) = \frac{1}{\nu+1} D(\alpha) \left(\frac{\nu}{\nu+1}\right)^{a^\dagger a} D^\dagger(\alpha),$$

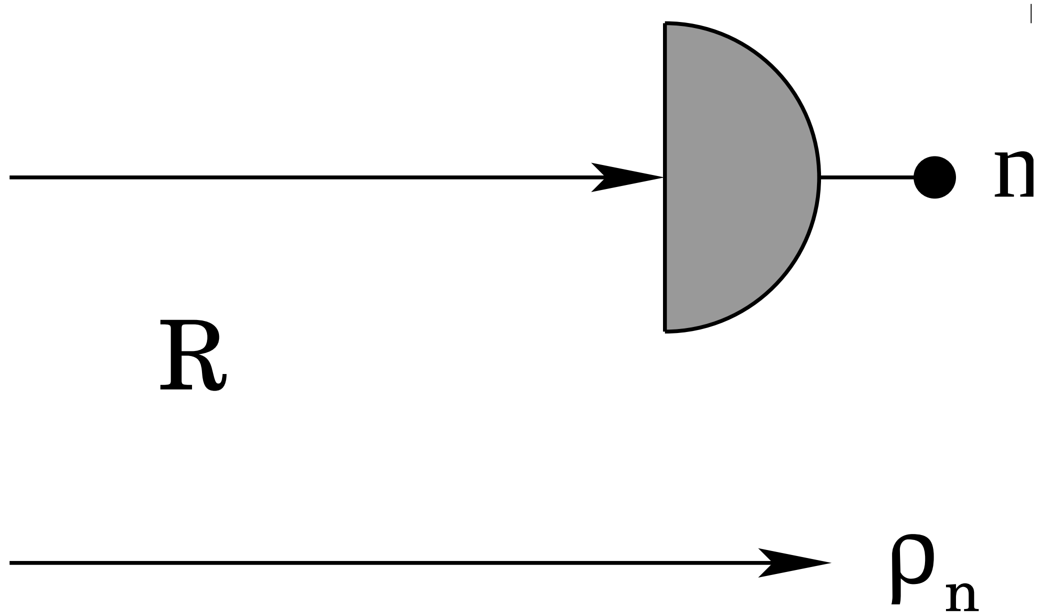
one obtains

$$\mathcal{N}_\nu^{-1}(|\alpha\rangle\langle\alpha|) = \frac{1}{1-\nu} D(\alpha) \left(1-\nu^{-1}\right)^{-a^\dagger a} D^\dagger(\alpha),$$

- which has convergence radius $\nu \leq \frac{1}{2}$, which is the well known bound for Gaussian noise for the quantum tomographic reconstruction for coherent-state and Fock representations.[2]
- Therefore, we say that the state is *formally* faithful, however, we are constrained to representations which are analytical for the inverse map $\mathcal{R}^{-1}$.

[2] G. M. D'Ariano, and N. Sterpi, J. Mod. Optics **44** 2227 (1997)

Tomography of POVM's



The conditioned state is given by

$$\rho_n = \frac{\mathrm{Tr}_1[(P_n \otimes I)R]}{\mathrm{Tr}[(P_n \otimes I)R]},$$

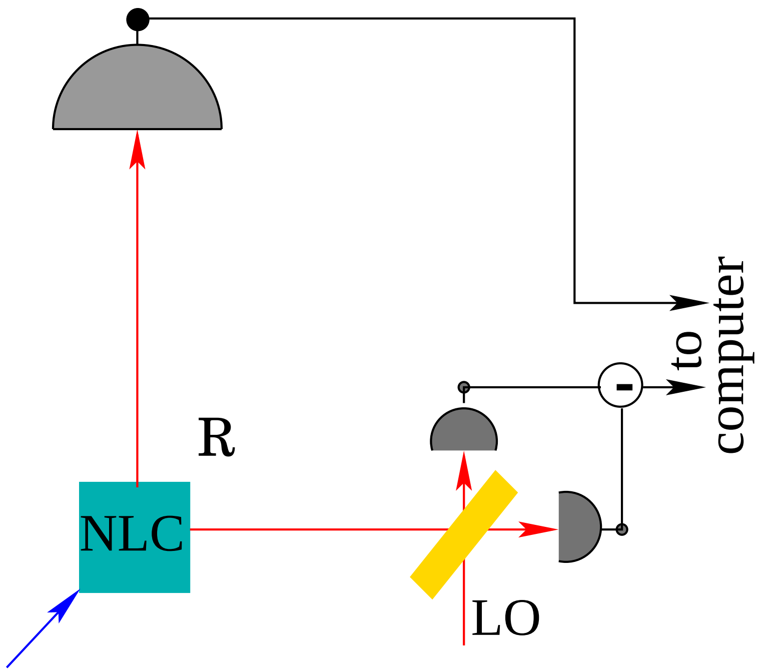
namely

$$\rho_n = \frac{\mathcal{R}(P_n^\tau)}{\mathrm{Tr}[\mathcal{R}(P_n^\tau)]},$$

where

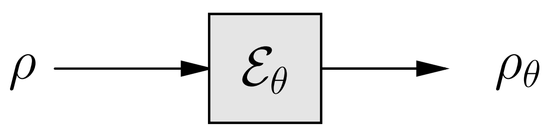
$$\mathcal{M}(\rho) = \sum_k A_k^\tau \rho A_k^*, \quad R = \sum_k |A_k\rangle\!\rangle\langle\!\langle A_k|.$$

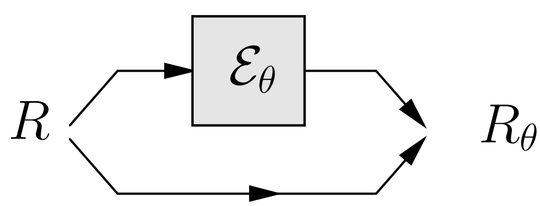
Absolute characterization of a photodetector



Discrimination among quantum operations

- General measurement problem: **optimal estimation of parameters of a quantum operation**.
- Optimization over:
  1. the detection scheme
  2. the input state
- The use of an entangled input state R is considered, with the unknown transformation $\mathcal{E}_\theta$ acting locally only on one side of the entangled state: $R \to R_\theta = \mathcal{E}_\theta \otimes \mathcal{I}(R)$.





- Result: **the entangled configuration is always better**, for either precision or stability of the measurement. This is due to the fact that the input entangled state is equivalent to many input states in "quantum parallel".

Precision increases with the dimension of the input space

- Instructive example: discrimination among the four Pauli matrices

$$\sigma_0 \equiv I\,, \sigma_1 \equiv \sigma_x\,, \sigma_2 \equiv \sigma_y\,, \sigma_3 \equiv \sigma_z.$$

- With a state $|\psi\rangle \in \mathbf{C}^2$ we always obtain four linearly dependent states $\sigma_j|\psi\rangle$ (we can discriminate exactly at most between two of them).

- On the contrary, if we apply the four matrices to the maximally entangled input state $\frac{1}{\sqrt{2}}|I\rangle\!\rangle$ we obtain the four Bell orthogonal states

$$(\sigma_j \otimes I)\frac{1}{\sqrt{2}}|I\rangle\!\rangle \equiv \frac{1}{\sqrt{2}}|\sigma_j\rangle\!\rangle.$$

namely we obtain perfect discrimination.

Precision increases with the dimension of the input space

- A less trivial example: estimating the displacement of a harmonic oscillator in the phase space, i. e. the parameter $\alpha \in \mathbb{C}$ of the transformation

$$\rho \longrightarrow \rho_\alpha = D(\alpha)\rho D^\dagger(\alpha),$$

where $D(\alpha)$ is the displacement operator for annihilation and creation operators a and $a^\dagger$

$$D(\alpha) = \exp(\alpha a^\dagger - \overline{\alpha} a).$$

- For unentangled ρ, an estimation of α isotropic on $\mathbb{C}$ is equivalent to a **optimal joint measurement of position and momentum**, which, as well known, is affected by a **unavoidable minimum noise of 3dB**.

- The optimal state (for fixed minimum energy) is the vacuum, and the corresponding conditional probability of measuring z given α is

$$p(z|\alpha) = \pi^{-1} \exp[-|z-\alpha|^2].$$

- Consider the same estimation with $D(\alpha)$ acting on the entangled state

$$|\Psi\rangle\!\rangle = \sqrt{1-|\xi|^2} \sum_{n=0}^{\infty} \xi^n |n\rangle \otimes |n\rangle \ ,$$

measuring the current $Z = a \otimes I - I \otimes a^\dagger$.

- We obtain

$$p(z|\alpha) = (\pi\Delta^2)^{-1} \exp[-\Delta^{-2}|z-\alpha|^2],$$

- with variance $\Delta^2 = \frac{1-|\xi|}{1+|\xi|}$ that, in principle, can be decreased at will (the state $|\Psi\rangle\!\rangle$ approaches an eigenstate $|D(z)\rangle\!\rangle$ of $Z = a \otimes I - I \otimes a^\dagger$.

Precision increases with the dimension of the input space

- The measurement of the current $Z = a \otimes I - I \otimes a^\dagger$ is achieved by a heterodyne detector

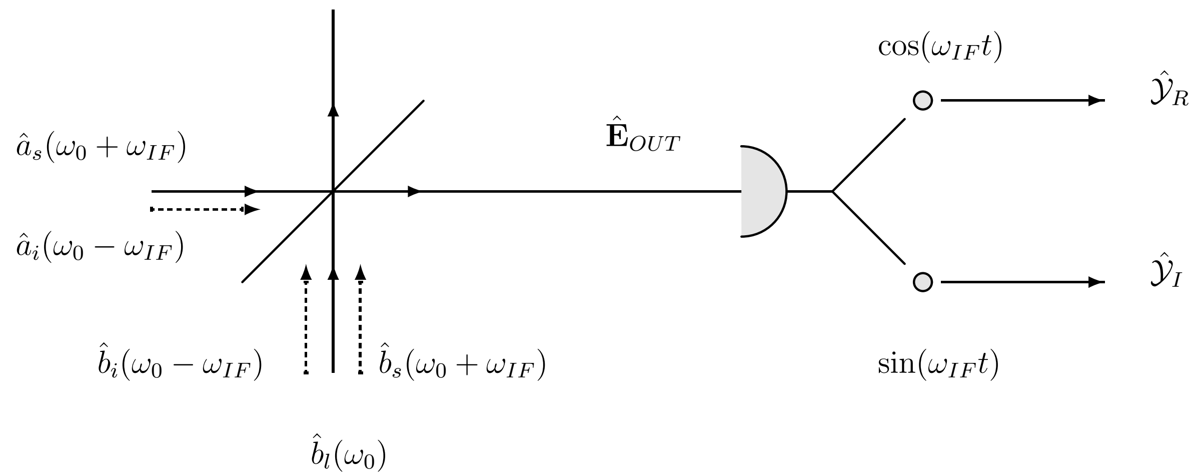


Figure 7: Scheme of the heterodyne detector and the relevant field modes involved in the measurement. Dashed lines denote vacuum states.

- Measured photocurrent:

$$\hat{I}_{OUT}(\omega_{IF}) = \int d\omega \hat{E}^-_{OUT}(\omega + \omega_{IF}) \hat{E}^+_{OUT}(\omega) \ .$$

- Reduced complex current $\hat{\mathcal{Y}}$ in the limit of transparent beam splitter and strong LO

$$\boxed{\hat{I}_{OUT} \propto a_s + a_i^\dagger}$$

- The heterodyne achieves the ideal measurement of the complex field a (here the *signal mode* a_s)
- Measuring a is equivalent to the joint measurement of $\hat{P}$ and $\hat{Q}$.
- The image-band mode a_i is vacuum, and its fluctuations are responsible for an additional 3dB of noise (minimum noise for joint measurement).

Precision increases with the dimension of the input space

- Optimal scheme for estimating a displacement using heterodyne and downconversion of vacuum in the signal and image bands.

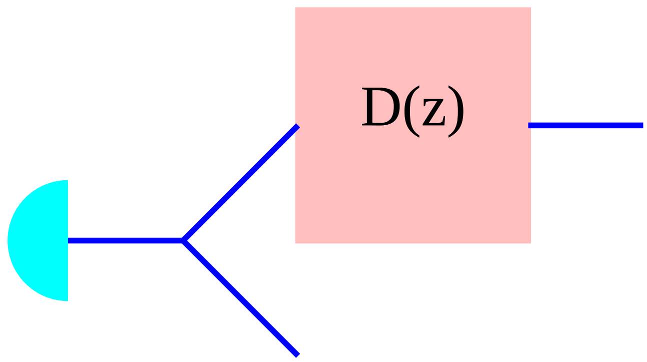


Figure 8: Conventional scheme with 3dB added noise

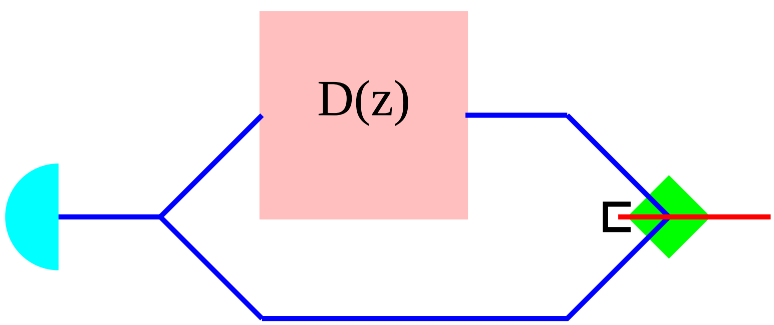


Figure 9: Unconventional scheme with vanishingly small noise

Precision increases with the dimension of the input space

- More generally, consider the discrimination among a group of unitaries $\{U_g\}$, $g \in \mathbf{G}$ that form a (projective) representation of the group $\mathbf{G}$, i. e. $U_g U_h = \omega(g,h) U_{gh}$, where $\omega(g,h)$ is a suitable phase [for simplicity let us consider an irreducible representation].

- Trace identity (from the Schur's lemma)

$$[U_g O U_g^\dagger]_{\mathbf{G}} = \mathrm{Tr}[O] I,$$

where $[f(g)]$ denotes the group averaging

$$[f(g)]_{\mathbf{G}} \doteq \frac{d}{|\mathbf{G}|} \sum_{g \in \mathbf{G}} f(g).$$

- For a general input state $|E\rangle\!\rangle \in \mathsf{H} \otimes \mathsf{H}$, the Hilbert space H_{out} spanned by the output states is the support of the operator

$$O = [|\Psi_g\rangle\!\rangle\langle\!\langle\Psi_g|]_{\mathbf{G}} \equiv (E^\dagger E)^\tau \otimes I,$$

with $\Psi_g = U_g E$. Therefore, $\dim(\mathsf{H}_{out}) = d \times \mathrm{rank}(E) \equiv$ Schmidt number of $|E\rangle\!\rangle$.

- Holevo bound $\chi = S([|\Psi_g\rangle\!\rangle\langle\!\langle\Psi_g|]_{\mathbf{G}}) - [S(|\Psi_g\rangle\!\rangle\langle\!\langle\Psi_g|)]_{\mathbf{G}}$ for the information accessible from the measurement, S denoting the von Neumann entropy $S(\rho) = -\mathrm{Tr}[\rho \ln \rho]$. One has

$$\chi = d^{-1} \ln d + S[E^\dagger E],$$

i. e. the bound is increased exactly of the amount of entanglement $S[E^\dagger E]$ of the input state.

Measurements in the presence of noise

- What happens if the estimation is performed in the presence of noise?
- Instructive example: estimating the displacement of a harmonic oscillator in the phase space in the presence of Gaussian displacement noise with ν mean thermal photon number

$$\rho \rightarrow \Gamma_\nu(\rho) \doteq \int_{\mathbb{C}} \frac{\mathrm{d}^2\gamma}{\pi\nu} e^{-\frac{|\gamma|^2}{\nu}} D(\gamma)\rho D^\dagger(\gamma).$$

- Noise particularly simple, since one has

$$\Gamma_\nu \Gamma_\mu = \Gamma_{\nu+\mu},$$

$$\Gamma_\nu[D(\alpha)\rho D^\dagger(\alpha)] = D(\alpha)\Gamma_\nu(\rho)D^\dagger(\alpha).$$

- If the measurement is made on the entangled state $|\Psi\rangle\!\rangle = (1 - |\xi|^2)^{\frac{1}{2}} \sum_{n=0}^{\infty} \xi^n |n\rangle \otimes |n\rangle$ one can easily derive a Gaussian conditional probability distribution with variance

$$\delta^2 = \Delta^2 + 2\nu_T,$$

  where $\Delta^2 = \frac{1-|\xi|}{1+|\xi|}$, ν_T is the total Gaussian displacement noise before and after the displacement $D(\alpha)$, and the noise is doubled since it is supposed equal on the two entangled Hilbert spaces.

- For unentangled input (vacuum), one has

$$\delta^2 = 1 + \nu_T.$$

- Conclusion: the entangled input is convenient below one thermal photon $\nu_T = 1$ of noise. This is exactly the threshold of noise above which the quantum capacity of the noisy channel vanishes.

Discrimination between two unitaries

- Optimal probability of error in the discrimination of the two output states $U_1|\psi\rangle$ and $U_2|\psi\rangle$ for any (also entangled) input state $|\psi\rangle$

$$P_E = \frac{1}{2}\left[1 - \sqrt{1 - 4p_1p_2|\langle\psi|U_2^\dagger U_1|\psi\rangle|^2}\right],$$

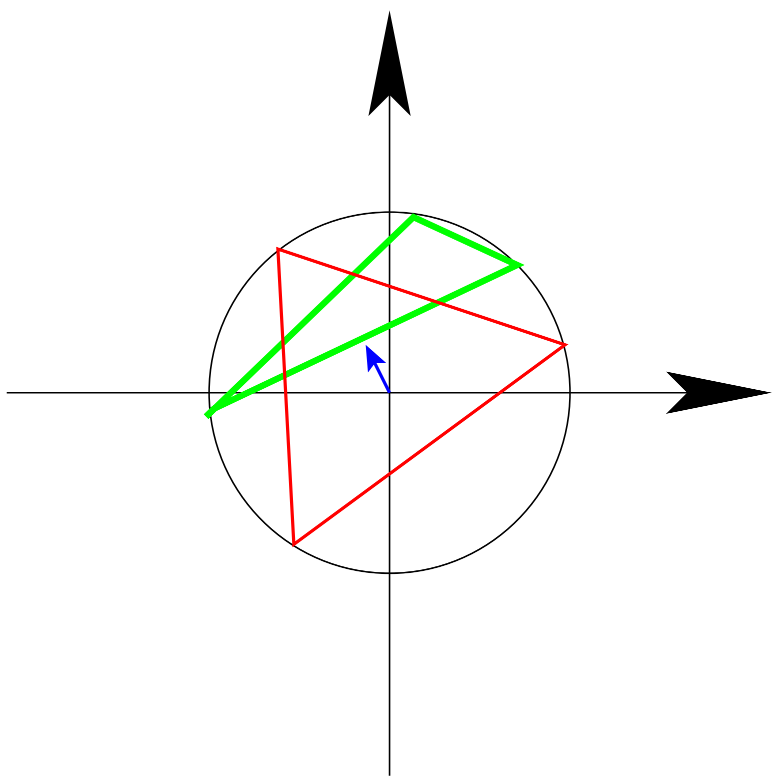
  p_1 and p_2 being the *a priori* probability of the two transformations (for simplicity, set $p_1 = p_2 = \frac{1}{2}$).

- Optimum input states $|\psi\rangle$ minimize the overlap $|\langle\psi|U_2^\dagger U_1|\psi\rangle|$.

- Minimum overlap is given by

$$\min_{||\psi||=1} |\langle\psi|U_2^\dagger U_1|\psi\rangle| = r(U_2^\dagger U_1),$$

- $r(W)$ denoting the distance between the origin of the complex plane and the poligon whose vertices are the eigenvalues of the unitary operator W.

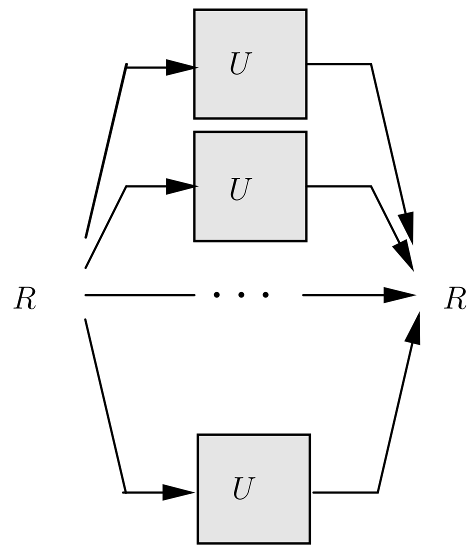


- Discrimination is perfect iff the poligon of the eigenvalues of $W = U_2^\dagger U_1$ encircles the origin.

Discrimination between two unitaries

- Since W and $W \otimes I$ have the same spectrum, an entangled input is useless.
- The situation changes dramatically if one has N copies of the unitary transformation $U = U_{1,2}$ to be determined.

- Use a N-partite entangled state as follows



- Now the spectrum of $W^{\otimes N}$ has angular spread $\Delta(W)$ that increases as $\Delta(W^{\otimes N}) = \min(N\Delta(W), 2\pi)$
- Conclusion: therefore, *the discrimination is always exact for sufficiently many uses* N! (compared with the case of *state* discrimination) [see also Acin].
- The above arguments could be extended to the case of multiple testing.

Improving the stability of the measurement

- When the discrimination is already optimized by a unentangled input, **an entangled state can still be better in achieving a more stable sensitivity**.
- A unentangled input is optimal in the covariant measurement for abelian group $\mathbb{G}$.
- Example: problem of distinguishing among displacements on a fixed direction of the phase space, say $D(x)$, with $x \in \mathbb{R}$.
- In this case one could use a squeezed input state

$$|x_0\rangle_s \doteq \exp\left[\frac{s}{2}((a^\dagger)^2 - a^2)\right] D(x_0)|0\rangle, \qquad s > 0,$$

squeezed in the direction of the "quadrature" $X = \frac{1}{2}(a^\dagger + a)$.

- A conditional Gaussian probability is obtained, with variance

$$\langle \Delta X^2 \rangle = \frac{1}{4} e^{-2s}$$

which can be narrowed at will by increasing the number $n_s = \sinh^2 s$ of squeezing photons.

- However, if the phase of the quadrature is slightly mismatched, and $X_\phi = \frac{1}{2}(a^\dagger e^{i\phi} + a e^{-i\phi})$ is measured instead, the variance becomes

$$\langle \Delta X_\phi^2 \rangle = \frac{1}{4}(e^{2s} \sin^2 \phi + e^{-2s} \cos^2 \phi),$$

- The **sensitivity is unstable**.
- Using the entangled input

$$\boxed{|\Psi\rangle\!\rangle = (1 - |\xi|^2)^{\frac{1}{2}} \sum_{n=0}^{\infty} \xi^n |n\rangle \otimes |n\rangle}$$

one obtains the Gaussian noise $\Delta^2 = \frac{1-|x|}{1+|x|}$ independently on ϕ by using $n = 2|x|^2/(1-|x|^2)$ downconverted photons!

CONCLUSIONS

- An entangled input state can be used for:
  1. determining experimentally the quantum operation of a device or a media via quantum tomography;
  2. improving the precision of the measurement of a parameter of a transformation;
  3. improving the precision in the presence of noise;
  4. achieving perfect discrimination between unitaries, when multiple copies are available;
  5. achieving a more stable precision for the measurement;
- The underlying mechanisms are:
  1. the entangled state works effectively as all possible input states in "quantum parallel";
  2. entanglement enlarges the dimension of H;
  3. increasing the dimension of the Hilbert space can be used to transform set of linearly dependent states $\{\rho\}$ into states that are linearly independent $\{\rho^{\otimes N}\}$.