Universal measuring devices and quantum calibration

Nottingham, School of Mathematical Sciences (December 17 2003)

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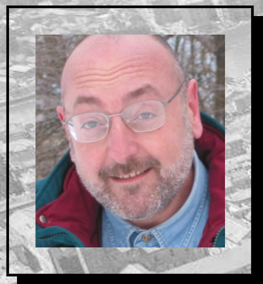
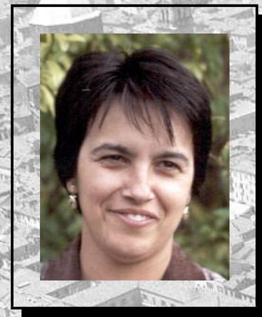


G. M. D'Ariano (full prof.)



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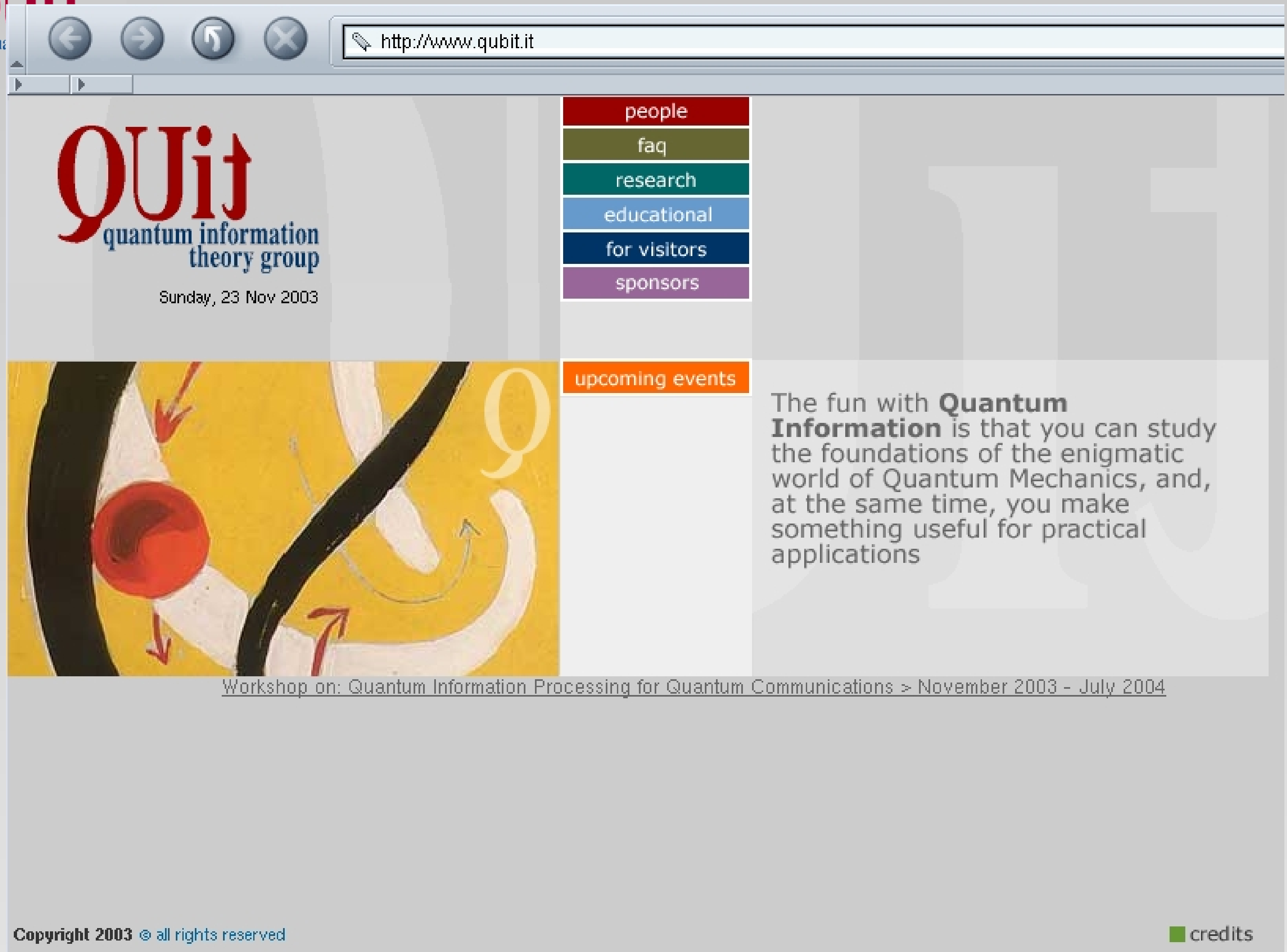
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3. **Absolute quantum calibration** (G. M. D. and P. Lo Presti)

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Universal quantum detectors

Definition:

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By a universal detector we can determine the expectation value $\langle O \rangle$ of an arbitrary operator O of a quantum system just by using a different *data-processing* for each O.

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$$\mathrm{Tr}[\rho O] = \sum_i f_i(\nu, O) \mathrm{Tr}[(\rho \otimes \nu)\Pi_i] \,, \tag{1}$$

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- In terms of the system only:

$$\mathrm{Tr}[\rho O] = \sum_i f_i(\nu, O)\mathrm{Tr}[\rho \Xi_i[\nu]], \qquad \Xi_i[\nu] \doteq \mathrm{Tr}_2[(I \otimes \nu)\Pi_i], \tag{2}$$

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the POVM $\{\Xi_i[\nu]\}$ is informationally complete.

[D'Ariano, Perinotti and Sacchi, quant-ph/0306025]

Notation for entangled states

- Hilbert-Schmidt isomorphism: $|\Psi\rangle\rangle \in \mathsf{H} \otimes \mathsf{K} \iff \Psi$ operator from K to H

$$|\Psi\rangle\rangle = \sum_{nm} \Psi_{nm} |n\rangle \otimes |m\rangle \qquad \iff \qquad \Psi = \sum_{nm} \Psi_{nm} |n\rangle\langle m|. \tag{3}$$

$$\langle\langle A|B\rangle\rangle \equiv \mathrm{Tr}[A^\dagger B]. \tag{4}$$

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- Partial trace rules

$$\begin{aligned} \mathrm{Tr}_{\mathsf{K}}[|A\rangle\!\rangle\langle\!\langle B|] &= AB^\dagger\,, \\ \mathrm{Tr}_{\mathsf{H}}[|A\rangle\!\rangle\langle\!\langle B|] &= (B^\dagger A)^\intercal\,, \end{aligned} \tag{5}$$

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- Multiplication rules (for fixed reference basis in the two Hilbert spaces):

$$(A \otimes B)|C\rangle\!\rangle \;=\; |AC\,B^\intercal\rangle\!\rangle, \tag{6}$$

$$|A\rangle\!\rangle \;\equiv\; (A \otimes I)|I\rangle\!\rangle \equiv (I \otimes A^\intercal)|I\rangle\!\rangle, \qquad |I\rangle\!\rangle = \sum_n |n\rangle \otimes |n\rangle, \tag{7}$$

$$(U \otimes U^*)|I\rangle\!\rangle \;=\; |I\rangle\!\rangle, \qquad U^* \doteq (U^\dagger)^\intercal. \tag{8}$$

Frames of operators

- A sequence of operators $\{\Xi_i\}$ is a frame for a Banach space of operators if there are constants $0 < a \leq b < +\infty$ s.t. for all operators A one has

$$a\|A\|^2 \leq \underbrace{\sum_i |\langle A, \Xi_i \rangle|^2}_{\text{Bessel series}} \leq b\|A\|^2. \tag{9}$$

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- The sequence of operators $\{\Xi_i\}$ is a frame iff the following operator on H $\otimes$ K is bounded and invertible (Hilbert-Schmidt operators)

$$F = \sum_i |\Xi_i\rangle\!\rangle\langle\!\langle\Xi_i| \,. \qquad \text{(frame operator)} \tag{10}$$

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$$F = \sum_i |\Xi_i\rangle\!\rangle\langle\!\langle\Xi_i| \,. \qquad \text{(frame operator)} \tag{10}$$

- Then, there exists a dual frame $\{\Theta_i\}$ such that every operator A can be expanded as follows

$$A = \sum_i \mathrm{Tr}[\Theta_i^\dagger A]\Xi_i \,. \tag{11}$$

Frames of operators

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- The completeness relation of the frame also reads:

$$E = \sum_i \Theta_i^\dagger \otimes \Xi_i \qquad E : \text{swap operator on } \mathsf{H} \otimes \mathsf{K} \tag{12}$$

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- Alternate dual frames:

$$|\Theta_i\rangle\!\rangle = F^{-1}|\Xi_i\rangle\!\rangle + |Y_i\rangle\!\rangle - \sum_j \langle\!\langle \Xi_j|F^{-1}|\Xi_i\rangle\!\rangle |Y_j\rangle\!\rangle \,, \tag{13}$$

Y_i arbitrary Bessel, and $F^{-1}|\Xi_i\rangle\!\rangle$ canonical dual frame.

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- For exact frames there is only the canonical dual frame. Alternate duals are useful for optimization.

Universal quantum detectors

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- The POVM $\{\Xi_i[\nu]\}$ is necessarily not orthogonal.

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Upon diagonalizing the POVM $\{\Pi_i\}$ on $\mathsf{H} \otimes \mathsf{K}$

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- It follows that $\{\Pi_i\}$ is universal iff both $\{\Psi_j^{(i)}\}$ and $\{\Xi_i[\nu]\}$ are operator frames.

Universal POVM's: the Bell case

$$\text{POVM on } \mathsf{H}\otimes\mathsf{H}: \quad \Pi_i = \frac{\alpha_i}{d}|U_i\rangle\!\rangle\langle\!\langle U_i|, \quad d = \dim(\mathsf{H}),\ \alpha_i > 0,\ U_i \text{ unitary.} \tag{19}$$

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- Special case: $\{U_i\}$ UIR of some group $\mathbf{G}$.

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- **Example:** projective UIR of abelian group: $\boxed{U_\alpha U_\beta U_\alpha^\dagger = e^{ic(\alpha,\beta)}U_\beta}$

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- **Example:** projective UIR of abelian group: $\boxed{U_\alpha U_\beta U_\alpha^\dagger = e^{ic(\alpha,\beta)} U_\beta}$
- One can prove that the Bell POVM is necessarily orthogonal and it is universal, with ancilla state ν satisfying $\mathrm{Tr}[U_\alpha^\dagger \nu^\intercal] \neq 0$ for all α.

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- One can prove that the Bell POVM is necessarily orthogonal and it is universal, with ancilla state ν satisfying $\mathrm{Tr}[U_\alpha^\dagger \nu^\intercal] \neq 0$ for all α.
- Dual set (unique) for data-processing:

$$\Theta_\alpha[\nu] = \frac{1}{d}\sum_{\beta=1}^{d^2}\frac{U_\beta e^{-ic(\beta,\alpha)}}{\mathrm{Tr}\left[U_\beta \nu^*\right]}\ . \tag{20}$$

[D'Ariano, Perinotti and Sacchi, quant-ph/0306025]

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- $\boxed{\textbf{Example:}}$ UIR of non abelian group $SU(d)$.

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$$F = \int \mathrm{d}\alpha \, (U_\alpha \otimes U_\alpha^*) \, |\nu^\intercal\rangle\!\rangle\langle\!\langle \nu^\intercal| \, (U_\alpha^\dagger \otimes U_\alpha^\intercal) = P + \frac{1}{a} P^\perp,$$
$$P \doteq \tfrac{1}{d} |I\rangle\!\rangle\langle\!\langle I|, \qquad a = \frac{d^2 - 1}{d \, \mathrm{Tr}[(\nu^\intercal)^2] - 1}, \tag{21}$$

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- Canonical dual frame

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$$\Theta_\alpha^0[\nu] = a\, U_\alpha\, \nu^\intercal U_\alpha^\dagger + b\, I\,, \qquad b = \frac{\mathrm{Tr}[(\nu^\intercal)^2] - d}{d\mathrm{Tr}[(\nu^\intercal)^2] - 1}. \tag{22}$$

[D'Ariano, Perinotti and Sacchi (ICSSUR, Puebla, Rinton Press) in press]

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One must have

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- The canonical dual frame minimizes the variance averaged over all pure states.
- The optimal ancilla state ν is pure.
- **Other examples:** $SU(2)$ UIR's on H with $\dim(\mathrm{H}) > 2$, ...

Universal POVM's: the separable case

For $\dim(\mathsf{K}) \geq \dim(\mathsf{H})^2$ one can obtain "separable" universal POVM's.

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- **Example:** observable operator frame on H

$$C(l) = \sum_k c_k(l) |c_k(l)\rangle\langle c_k(l)|, \qquad l = 1, 2, ..., L \geq \dim(\mathsf{H})^2 \,. \tag{25}$$

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[G. M. D'Ariano, Phys. Lett. A **268** *151 (2000)]*

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QUIT quantum information theory group

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$$\mathrm{d}\,P_g = \mathrm{Tr}_2[\mathrm{d}\,B_g(I \otimes \nu)] = \mathrm{d}\,g\,U_g \zeta U_g^\dagger, \qquad \zeta = V\nu^\intercal V^\dagger. \tag{30}$$

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- Bell measurement corresponding to the projective UIR of the Abelian group in d dimensions: $\mathbf{G} = \mathbf{Z}_d \times \mathbf{Z}_d$

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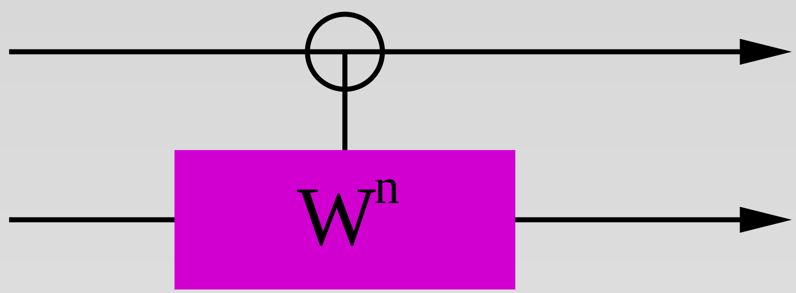

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- Nonorthogonal extremal POVM's are generally not connected by unitary transformations.

Convex structure of POVM's

Theorem 1 *The extremality of the POVM* $\mathbf{P} = \{P_n\}$ $n \in \mathsf{E} = \{1, 2, \ldots\}$ *is equivalent to the nonexistence of non trivial solutions* $\mathbf{D}$ *for the equation*

$$\boxed{\sum_n D_n = 0, \quad \mathsf{Supp}(D_n), \mathsf{Rng}(D_n) \subseteq \mathsf{Supp}(P_n). \qquad (34)}$$

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Theorem 2 (Parthasaraty) *A POVM* $\mathbf{P}$ *is extremal iff the operators* $|v_i^{(n)}\rangle\langle v_j^{(n)}|$ *are linearly independent, for all eigenvectors* $|v_j^{(n)}\rangle$ *of* P_n.

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This means that a POVM with too many elements (i. e. $N > d^2$) will be decomposable into several POVM's, each with less than d^2 non-vanishing elements.

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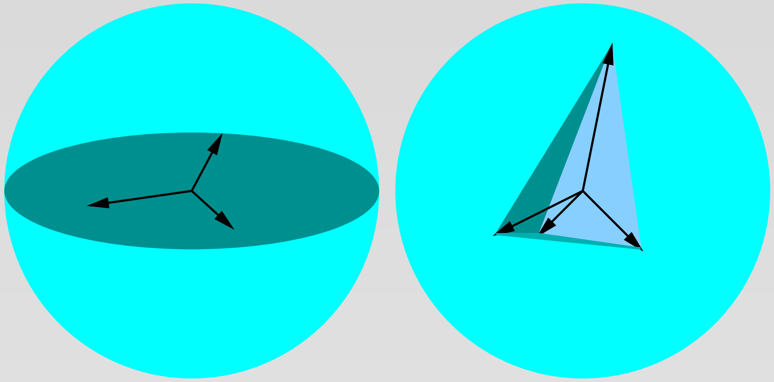

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- Approximate the observable $\mathbf{X}$ by a fixed programmable device

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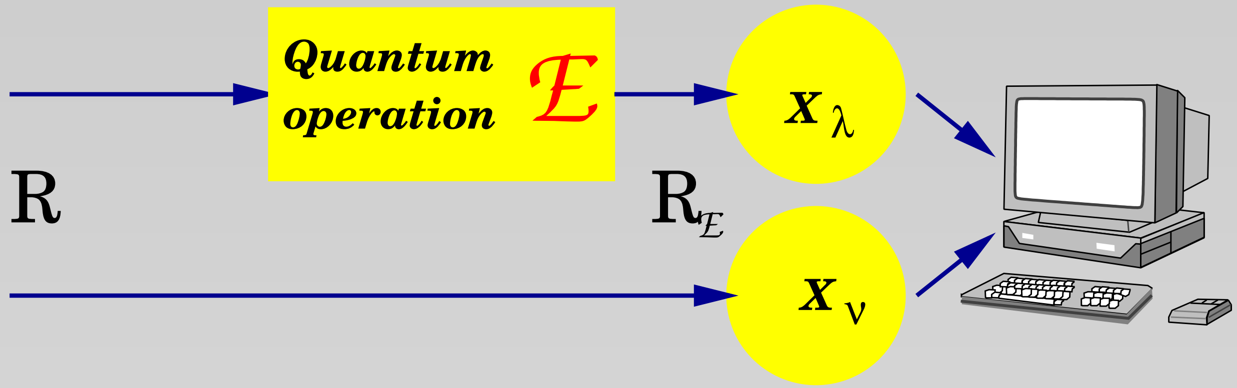
[G. M. D'Ariano and P. Perinotti, in progress]

Tomography of quantum operations



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[D'Ariano and Lo Presti, Phys. Rev. Lett. **86** *4195 (2001),Phys. Rev. Lett.* **91** *047902 (2003)]*

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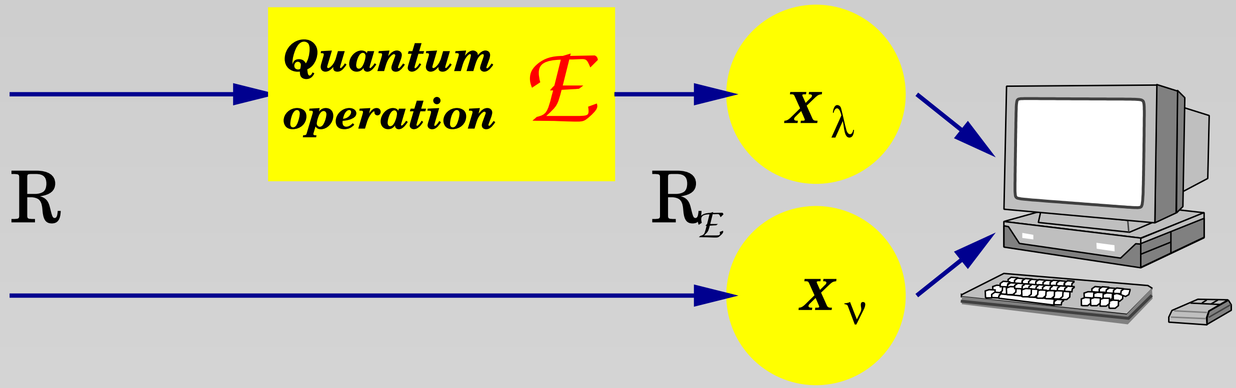

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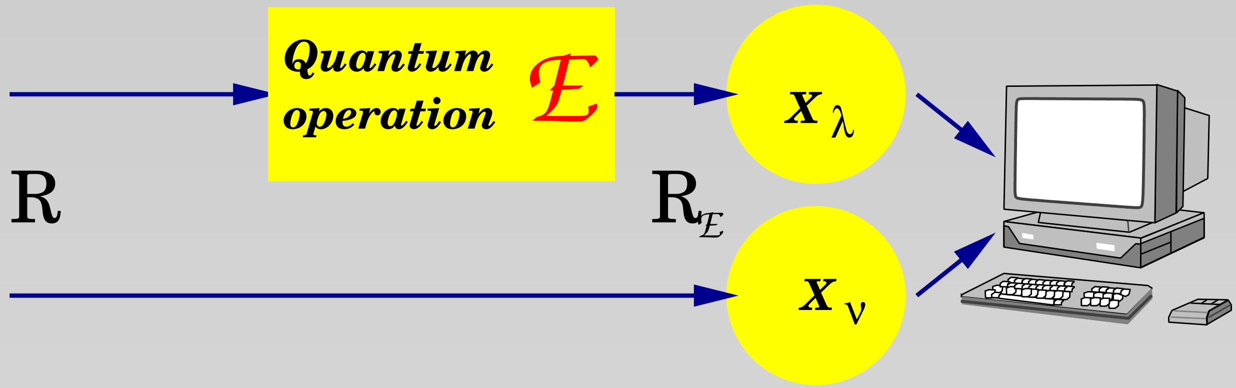

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For **faithful** input state R this is a 1-to-1correspondence between $R_{\mathscr{E}}$ and $\mathscr{E}$.
The quantum operation $\mathscr{E}$ is extracted from the output state as follows

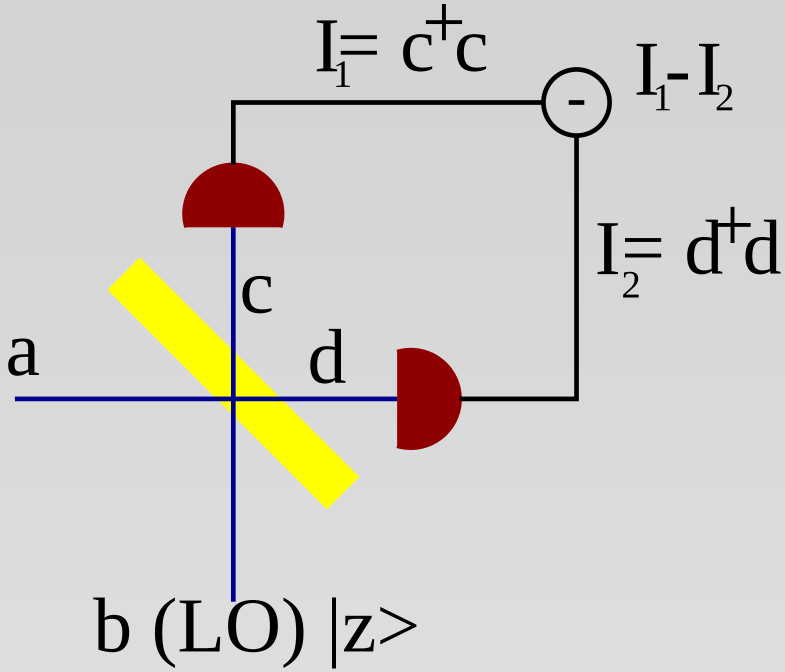
$$\mathscr{E}(\rho) = \mathrm{Tr}_2[(I \otimes \rho^\intercal)\mathscr{I} \otimes \mathscr{R}^{-1}(R_{\mathscr{E}})], \qquad \mathscr{R}(\rho) = \mathrm{Tr}_1[(\rho^\intercal \otimes I)R]. \tag{43}$$

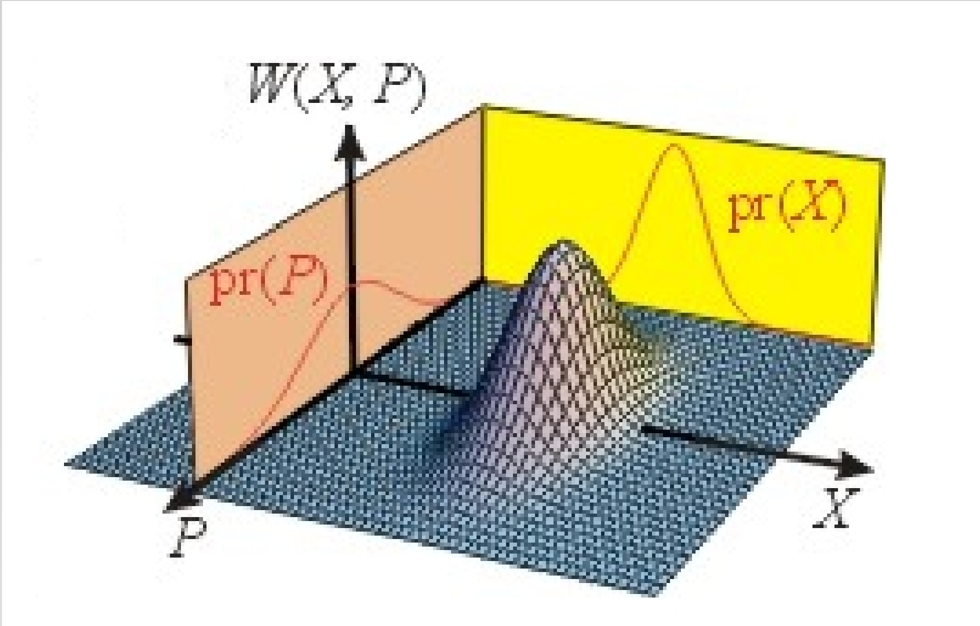
Homodyne tomography

- In quantum optics for each field mode a **quorum** $\equiv$ **{quadratures}**

$$\boxed{\mathsf{X}_\phi = \tfrac{1}{2}\left(a^\dagger e^{i\phi} + a e^{-i\phi}\right) \equiv Q\cos\phi + P\sin\phi.}$$

$$\langle H\rangle = \int_0^\pi \frac{\mathrm{d}\,\phi}{\pi}\langle E_H(X_\phi;\phi)\rangle\ , \qquad E_H(x;\phi) = \tfrac{1}{4}\int_{-\infty}^{+\infty}\mathrm{d}\,k\ |k|\mathrm{Tr}[H e^{ikX_\phi}]e^{-ikx}.$$





Pauli tomography

Pauli matrices I , σ_x , σ_y , σ_z orthonormal basis for the qubit operator space:

$$H = \frac{1}{2}[\boldsymbol{\sigma} \cdot \mathrm{Tr}(\boldsymbol{\sigma} H) + I\, \mathrm{Tr}(H)] \ .$$

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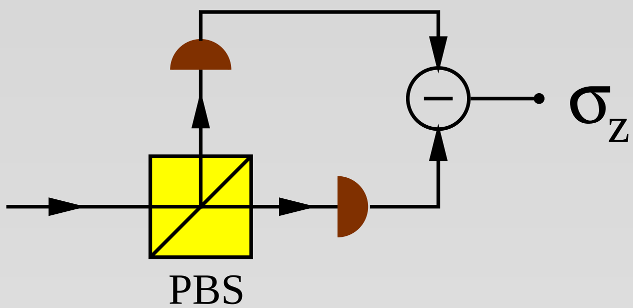
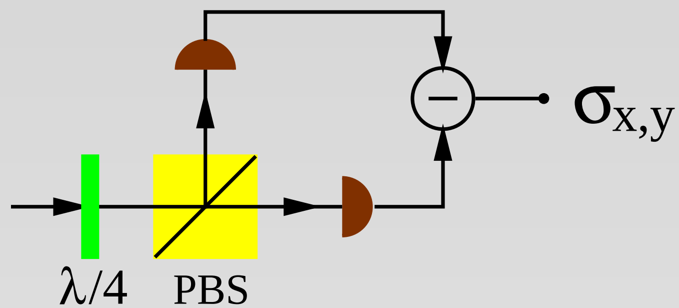
- In Quantum Optics the qubits are encoded on polarization of single photons:

$$\sigma_z = h^\dagger h - v^\dagger v,$$

$$|\uparrow\rangle \equiv |1\rangle_h |0\rangle_v, \qquad |\downarrow\rangle \equiv |0\rangle_h |1\rangle_v,$$





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- The set of faithful states R is *dense* within the set of all bipartite states.

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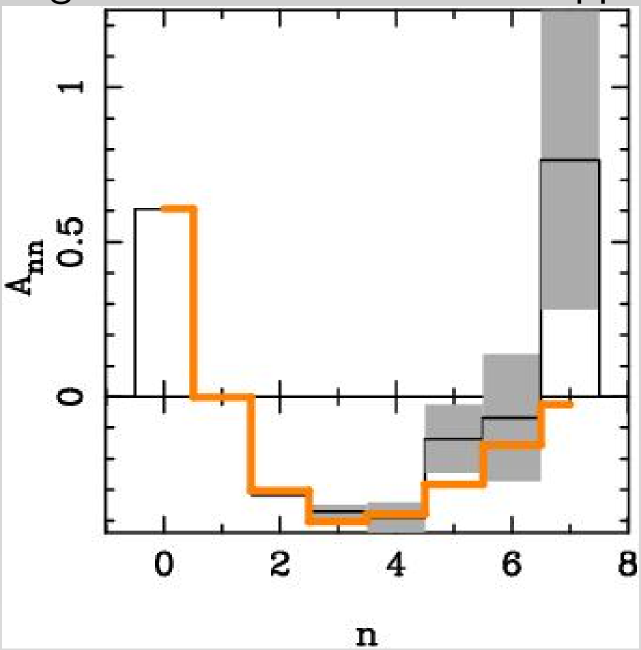

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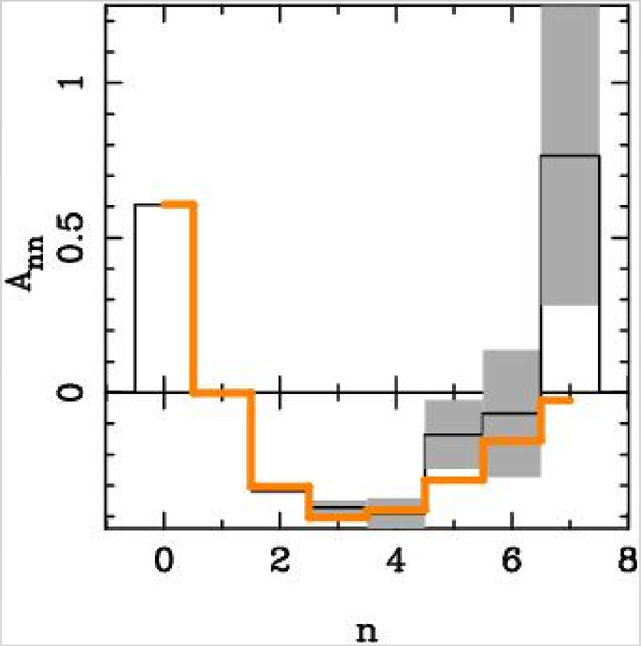

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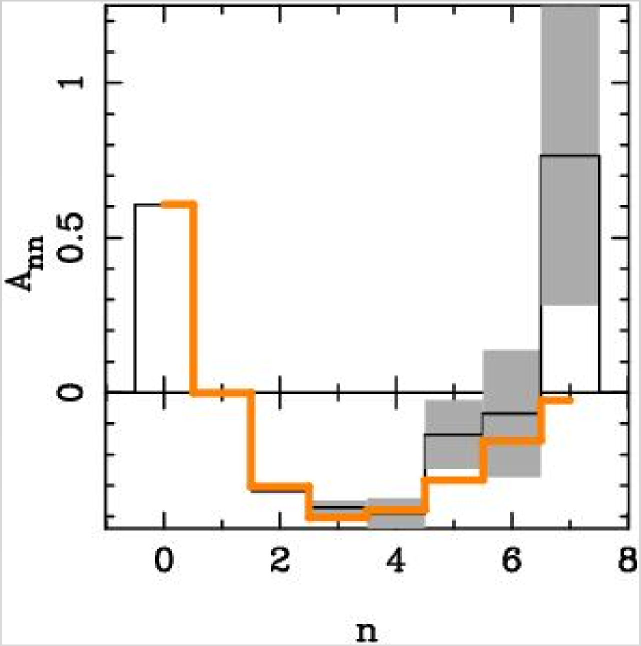

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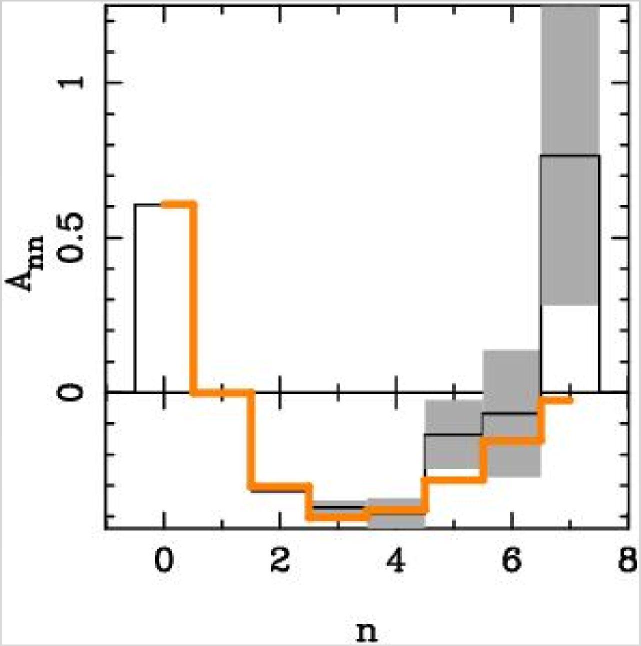
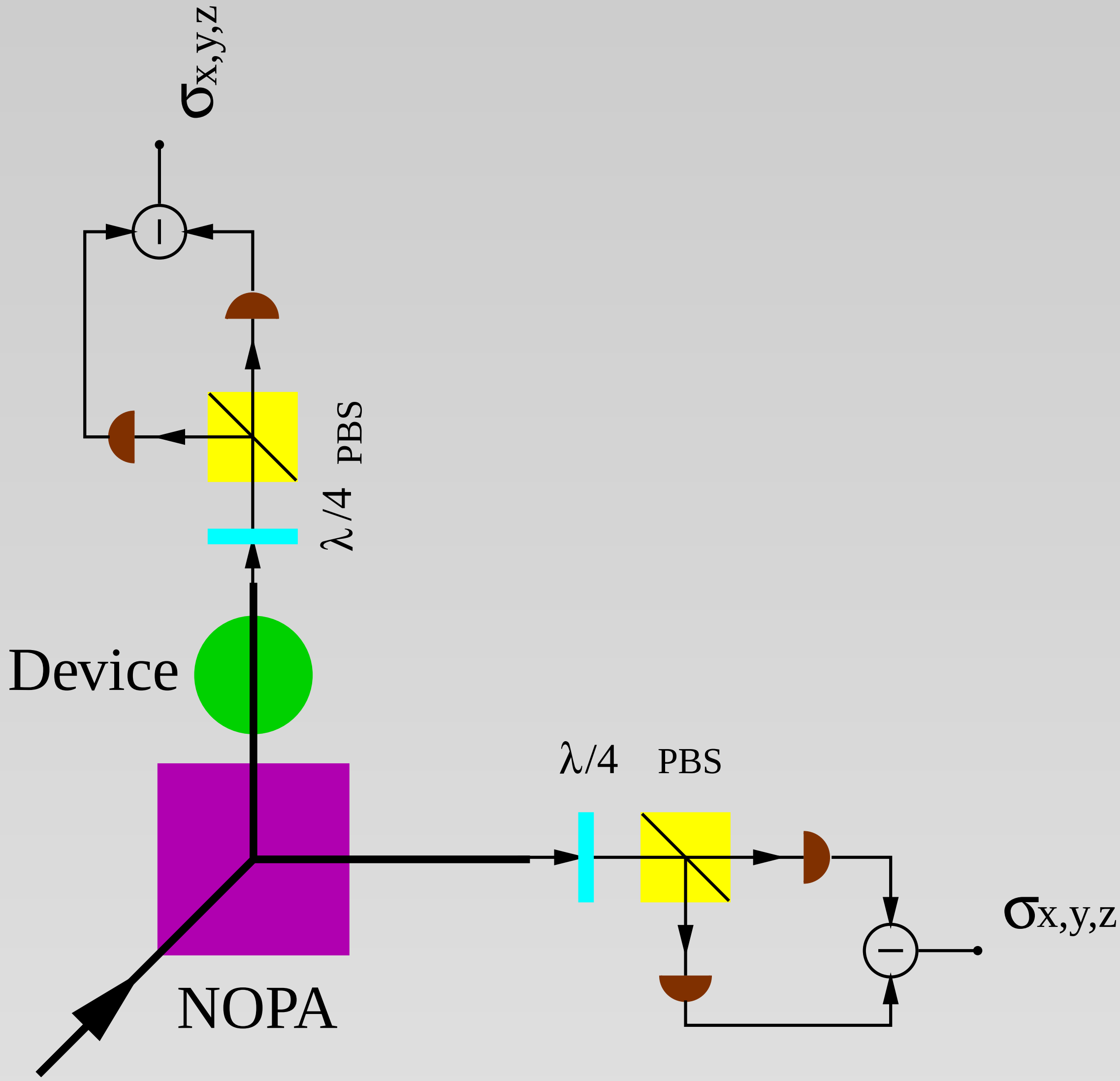


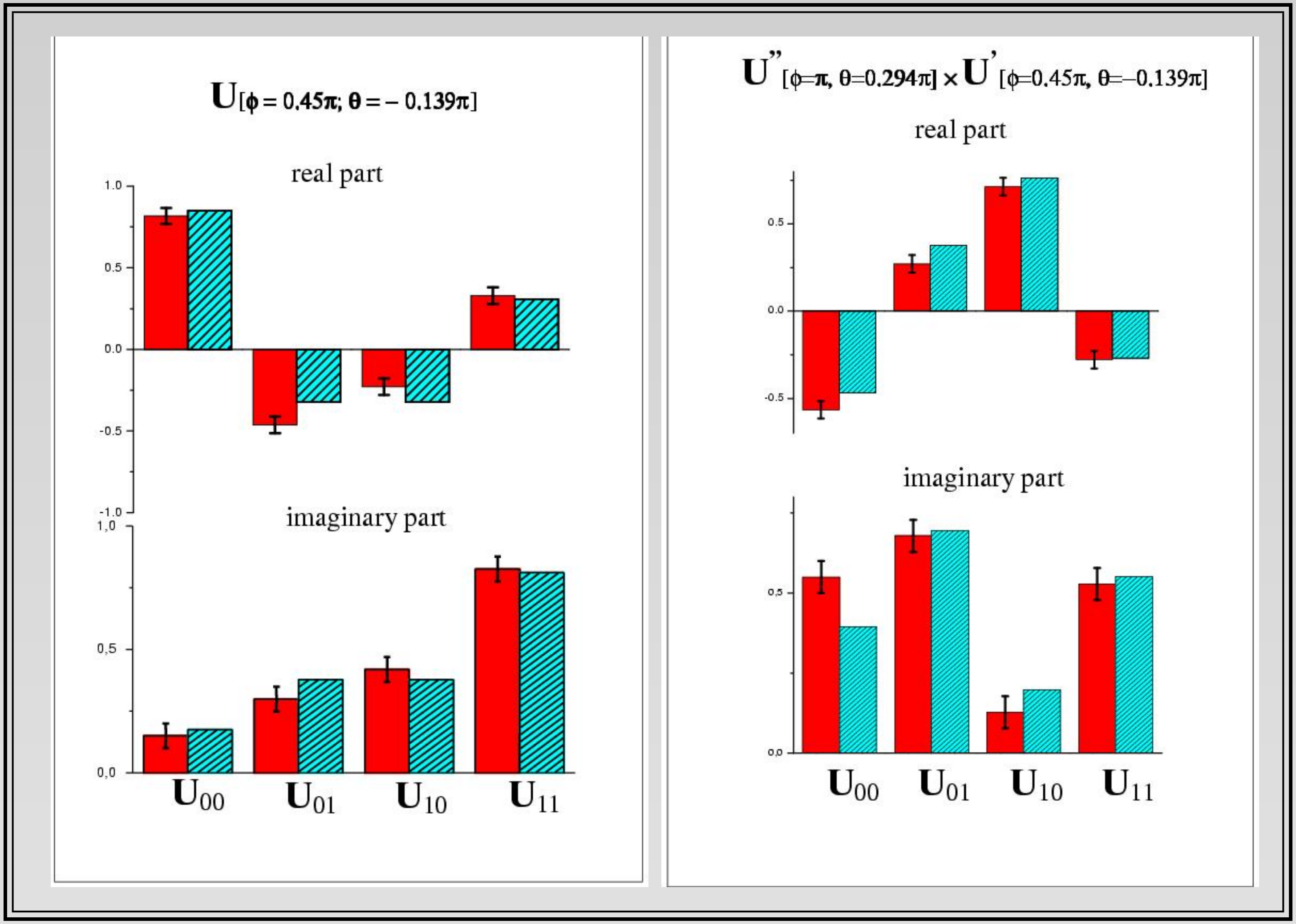
- Therefore, most mixed separable states are faithful! [e. g. Werner states are a. a. faithful].
- The most "efficient" states are the maximally entangled ones.
- For $d = \infty$ faithfulness depends also on the matrix representation [e. g. Gaussian displacement noise with $\overline{n} > \frac{1}{2}$].

Tomography of a single qubit quantum device

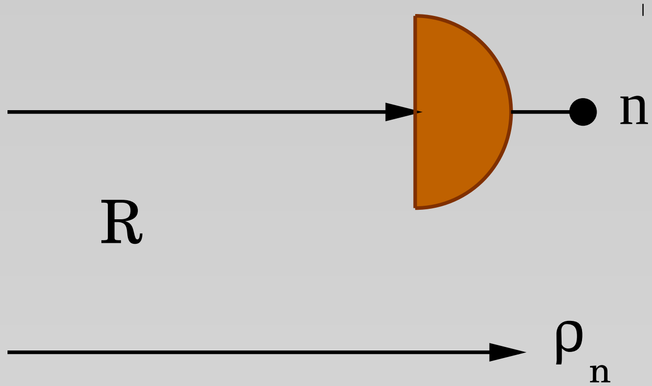


Tomography of a single qubit quantum device

Experiment performed in Roma La Sapienza



Absolute Quantum Calibration of a POVM



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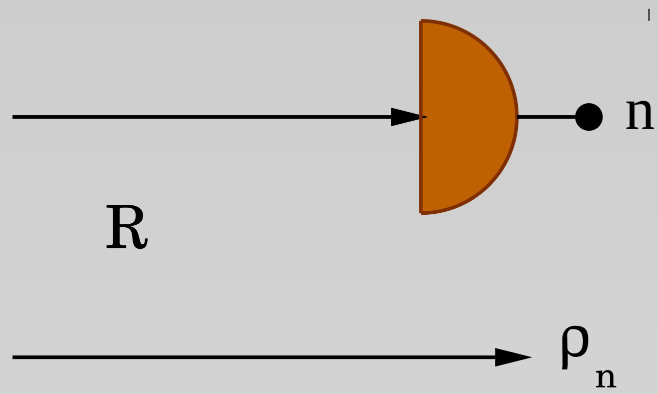

In terms of the POVM $\mathbf{P} \doteq \{P_n\}$ of the detector, the outcome n will occur with probability $p(n)$ corresponding to the conditioned state ρ_n given by

$$p(n) = \mathrm{Tr}[(P_n \otimes I)R], \qquad \rho_n = \frac{\mathrm{Tr}_1[(P_n \otimes I)R]}{\mathrm{Tr}[(P_n \otimes I)R]}, \tag{44}$$

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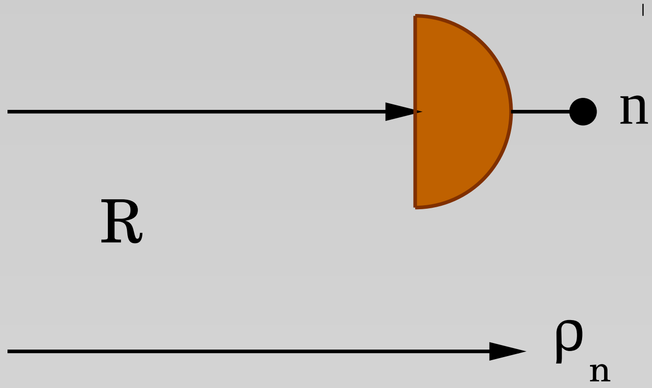

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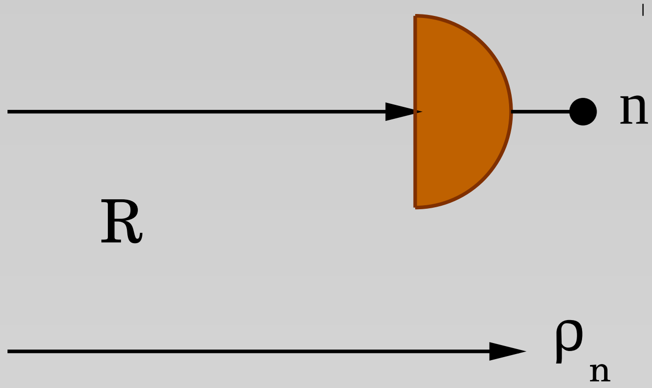
from which we can obtain the POVM as follows

$$P_n = p(n)[\mathscr{R}^{-1}(\rho_n)]^{\intercal}, \quad \mathscr{R}(\rho) = \mathrm{Tr}_1[(\rho^{\intercal} \otimes I)R]. \tag{45}$$

Absolute Quantum Calibration of Observable



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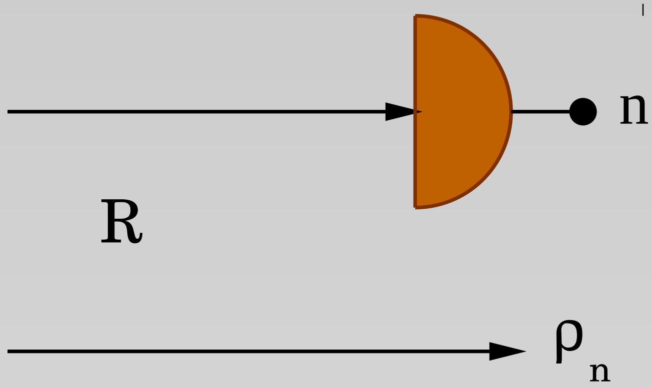

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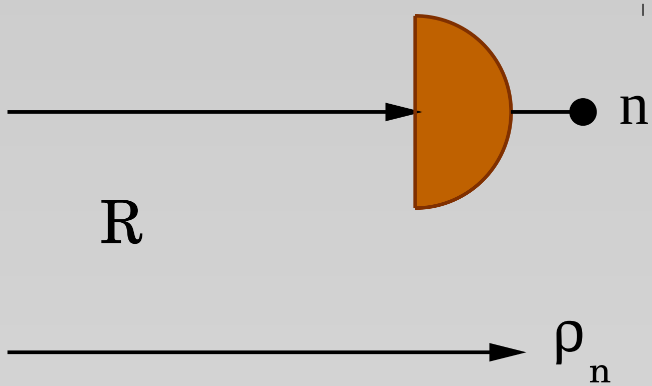

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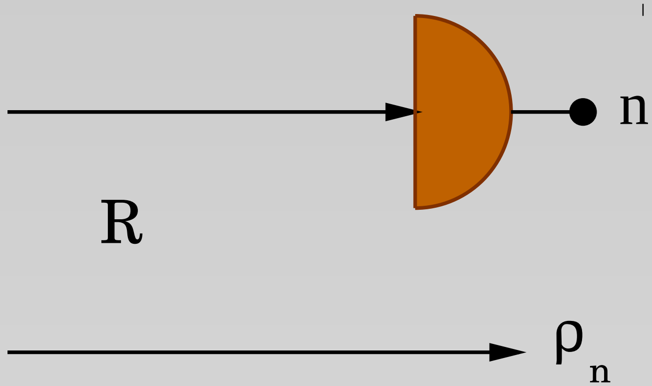
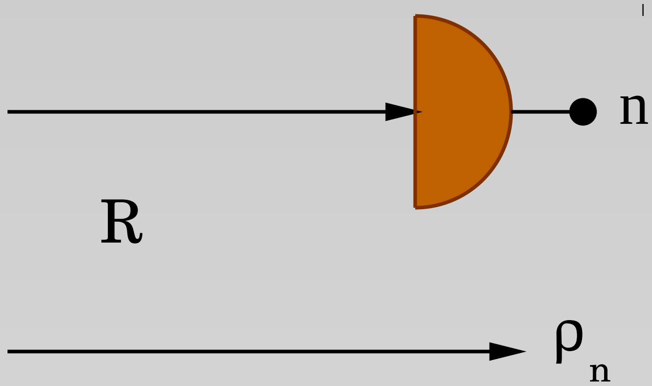
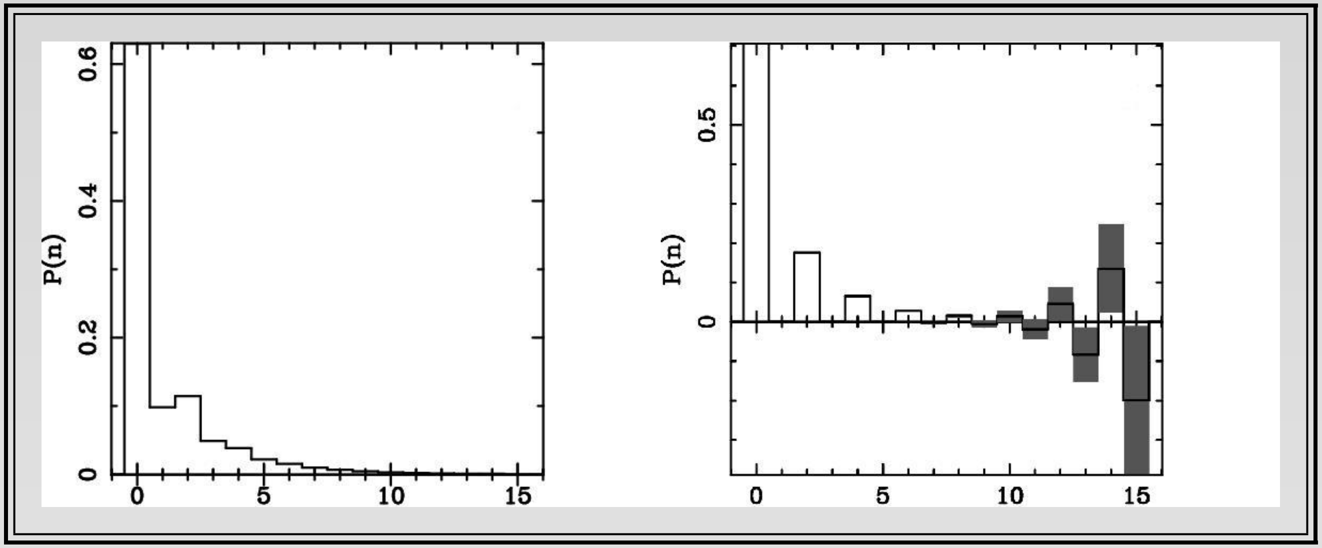


- From tomographic data one can recognize when the POVM is actually an "observable". This happens when the POVM is commutative.
- Then the POVM corresponds to any observable $K = \{|k\rangle\langle k|\}$ which commutes with $\{P_n\}$. From tomographic data one reconstructs the matrix elements $\langle k|P_n|k\rangle$ corresponding to the conditioned probability distribution $p(n|k) = \langle k|P_n|k\rangle$.

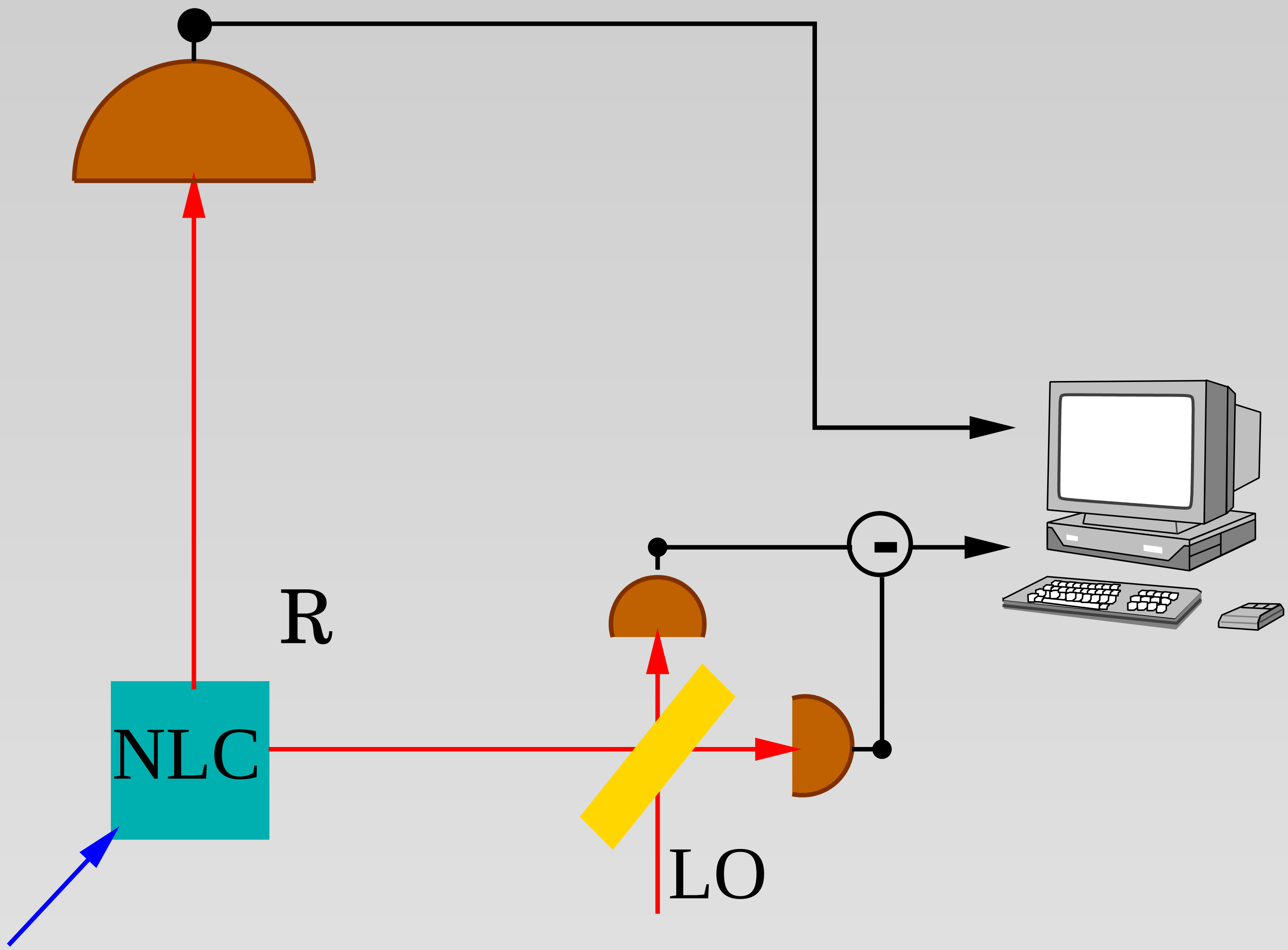
Absolute Quantum Calibration of Observable



- The conditioned probability $p(n|k)$ from the tomographic calibration will allow "unbiasing" the detector measurements.



Absolute calibration of a photodetector



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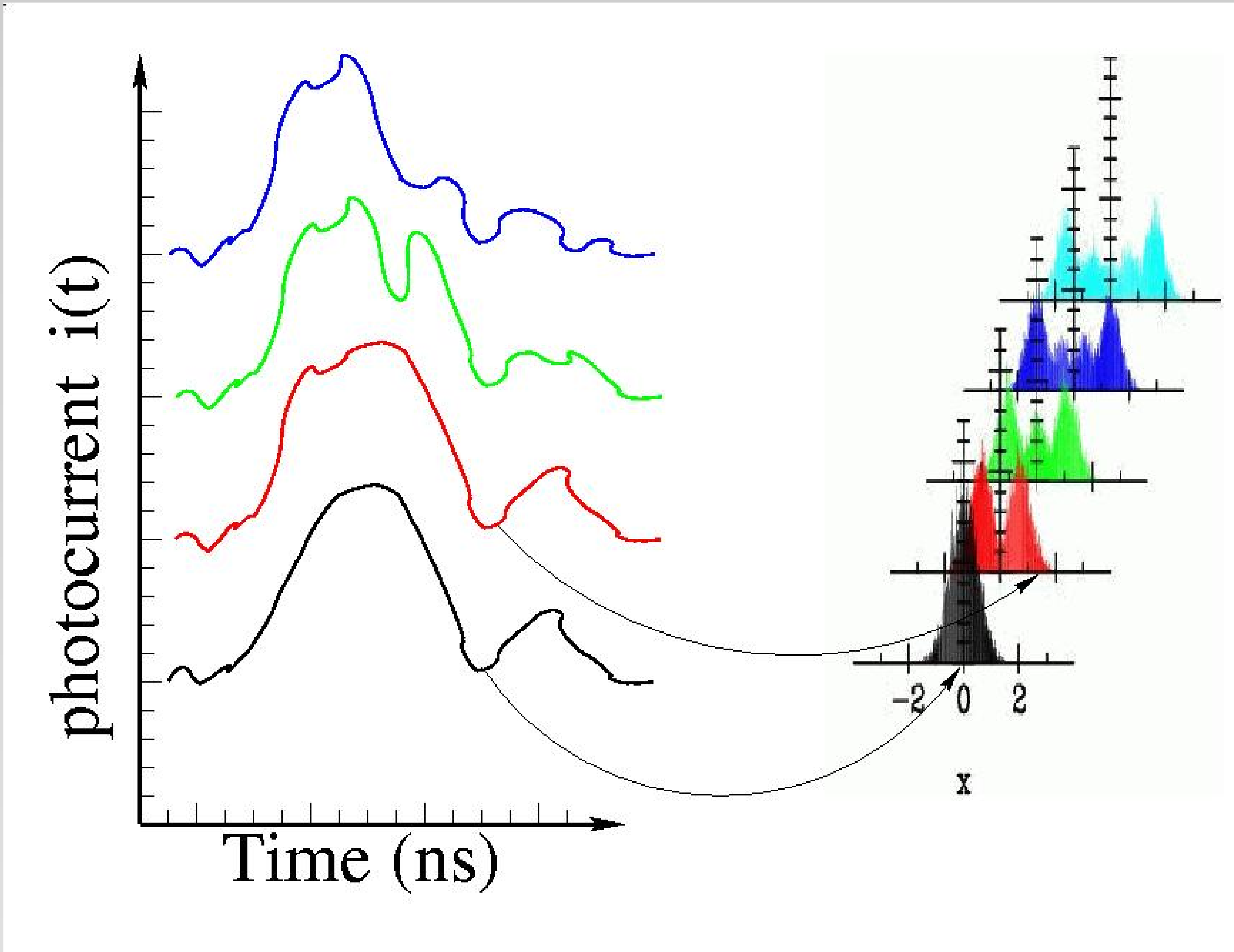

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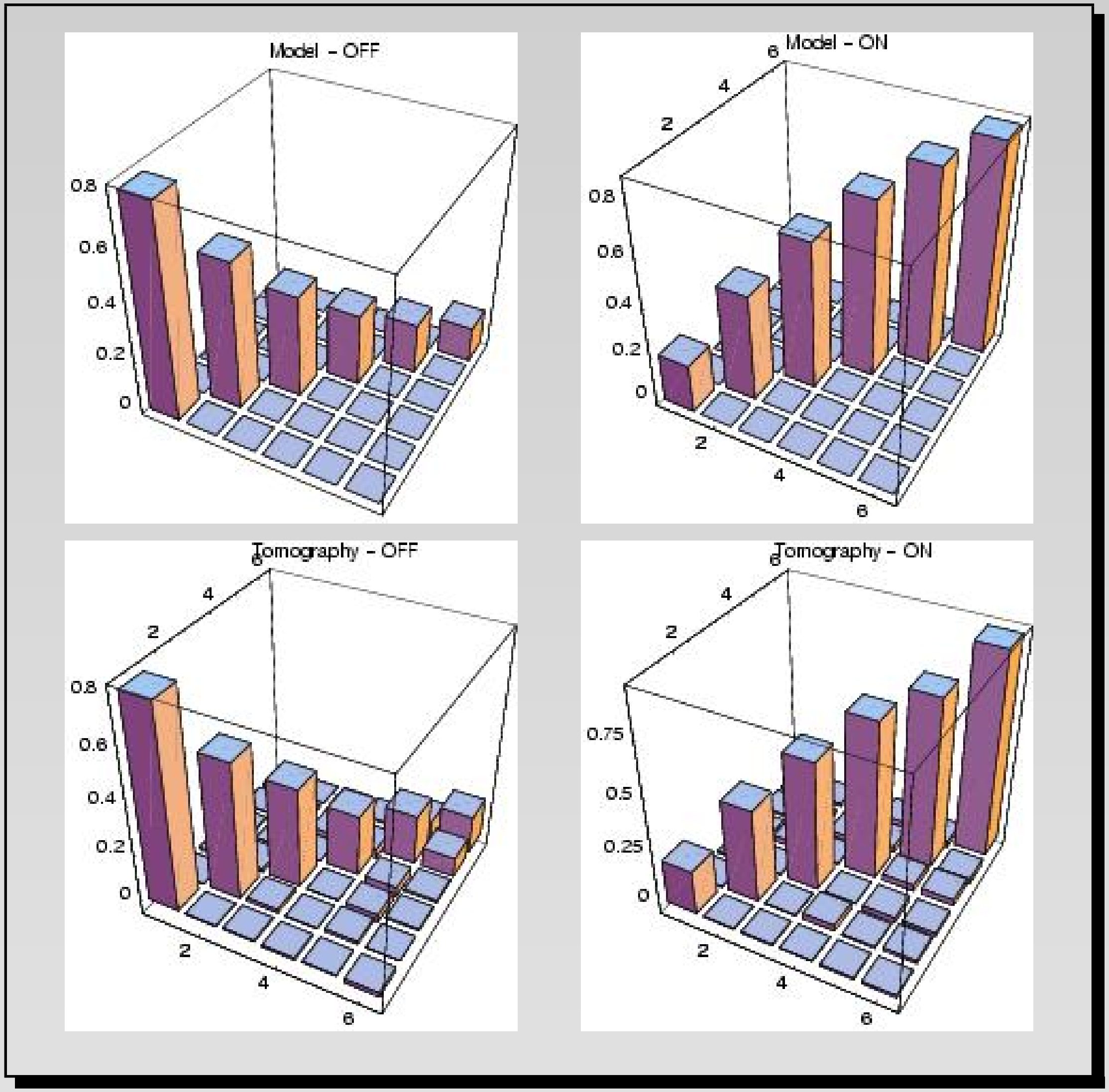

Homodyne calibration of a photodetector

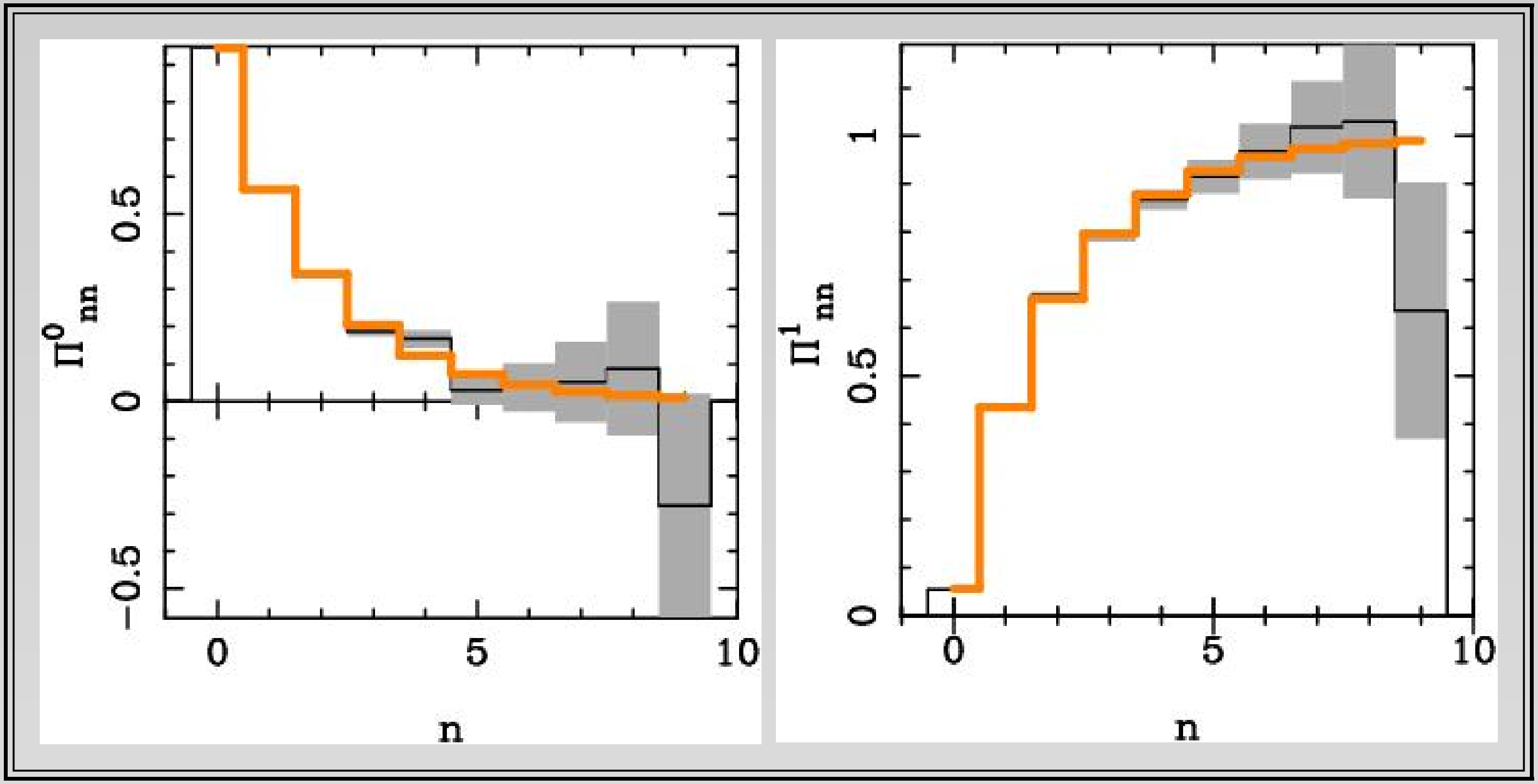

Figure 1: Homodyne tomography of an On/Off photo-detector with quantum efficiency $\eta = 0.4$ and thermal noise photon number $\nu = 0.1$. The reconstruction is obtained by pattern-function averaging of $1.5 \cdot 10^6$ data, for homodyne quantum efficiency $\eta = 0.9$ and twin beam thermal photon $\bar{n} = 3$.

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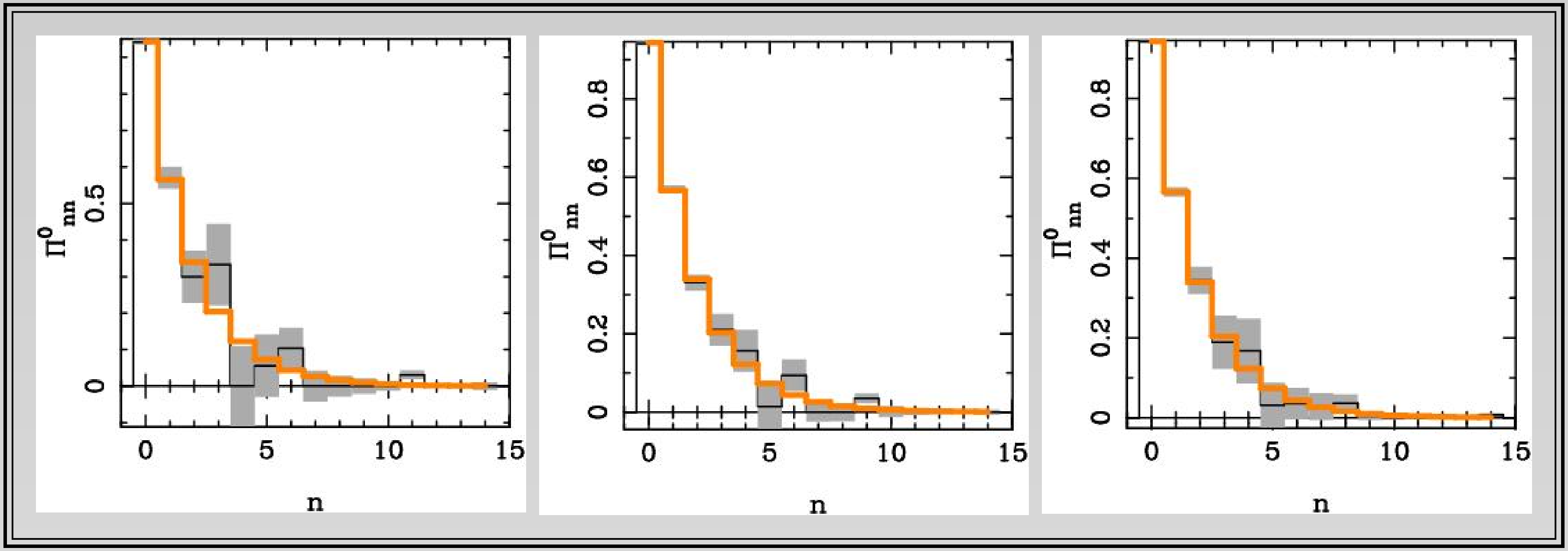
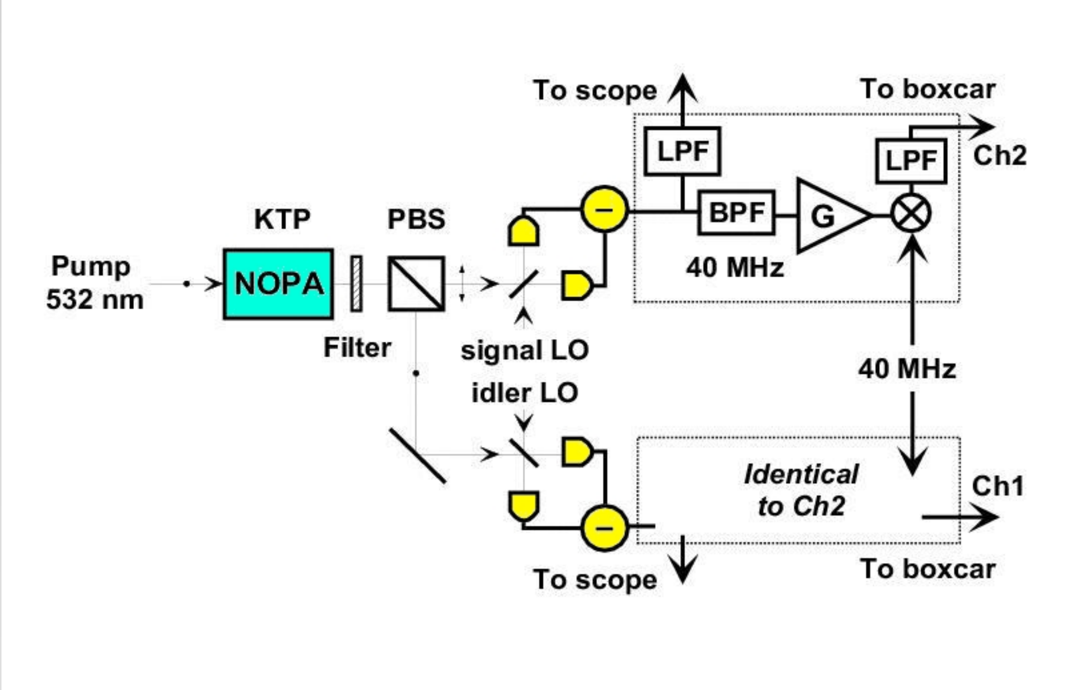


Figure 2: Homodyne tomography of an On/Off photodetector with quantum efficiency $\eta = 0.4$ and thermal noise photon number $\nu = 0.1$, with $\bar{n} = 3$ photons in the twin-beam. The ML estimation of the diagonal of the only Off POVM element are reported for different values of sample size N and homodyne quantum efficiency η_H. Left: $N = 10^5$, $\eta_H = 0.7$; Middle: $N = 10^4$, $\eta_H = 0.9$; Right: $N = 10^6$, $\eta_H = 0.7$.

NWU experiment on twin beam

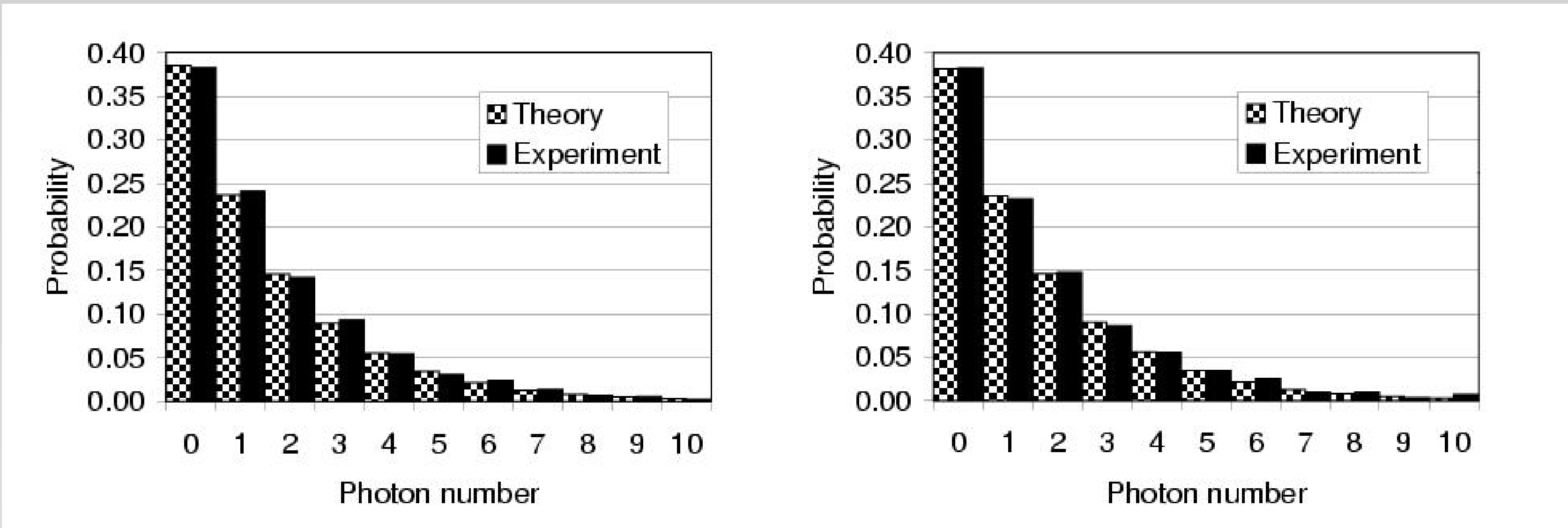
A schematic of the experimental setup. NOPA, non-degenerate optical parametric amplifier; LOs, local oscillators; PBS, polarizing beam splitter; LPFs, low-pass filters; BPF, band-pass filter; G, electronic amplifier. Electronics in the two channels are identical. The measured distributions exhibit up to 1.9 dB of quantum correlation between the signal and idler photon numbers, whereas the marginal distributions are thermal as expected for parametric fluorescence.



Measurement of the joint photon-number probability distribution for a twin-beam from nondegenerate downconversion

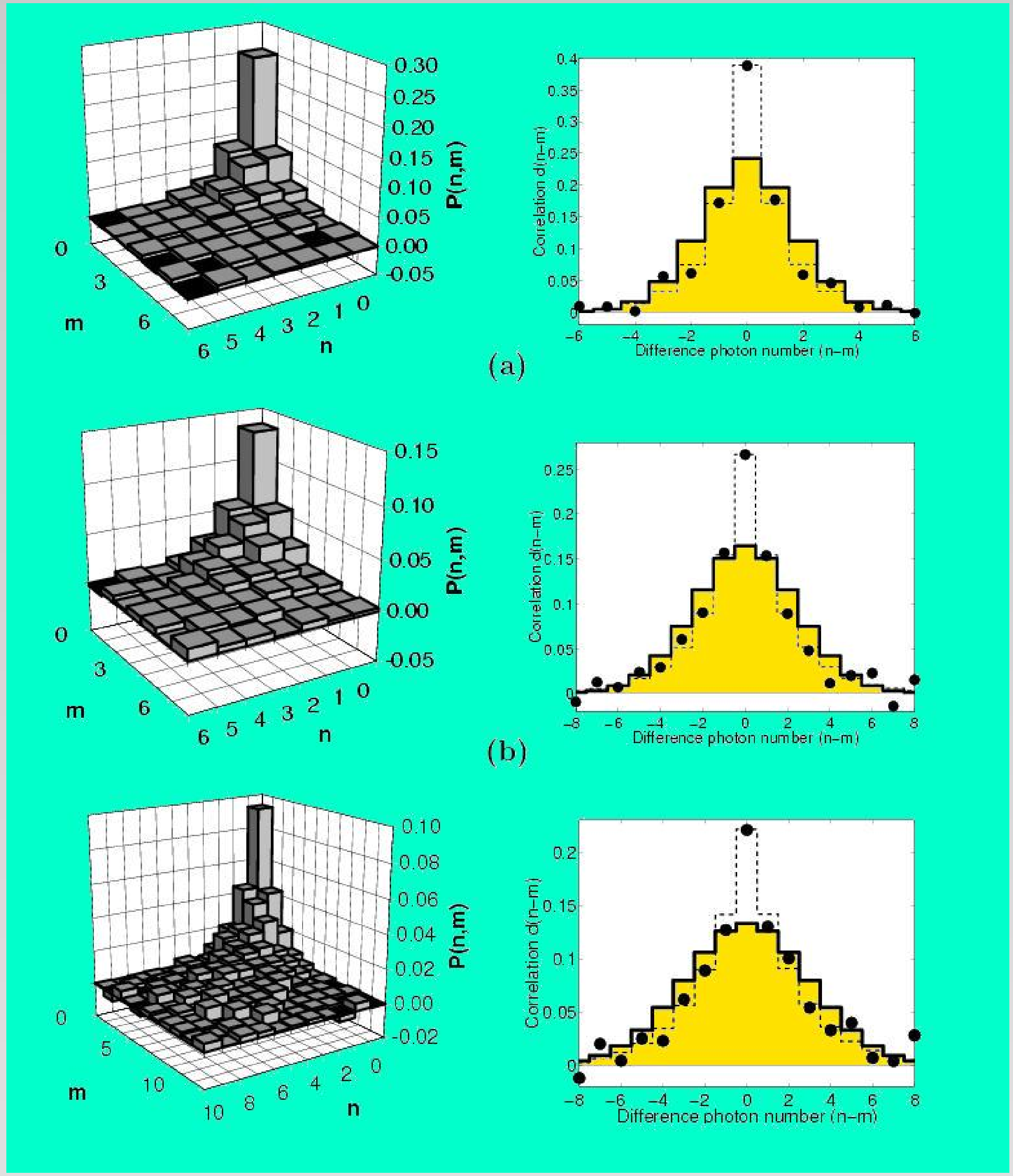
NWU experiment on twin beam

Marginal distributions for the signal and idler beams. Theoretical distributions for the same mean photon numbers are also shown [Phys. Rev. Lett. **84** 2354 (2000)].



Results

Left: Measured joint photon-number probability distributions for the twin-beam state. Right: Difference photon number distributions corresponding to the left graphs (filled circles, experimental data; solid lines, theoretical predictions; dashed lines, difference photon-number distributions for two independent coherent states with the same total mean number of photons and $\overline{n} = \overline{m}$.) (a) 400000 samples, $\overline{n} = \overline{m} = 1.5$, $N = 10$; (b) 240000 samples, $\overline{n} = 3.2$, $\overline{m} = 3.0$, $N = 18$; (c) 640000 samples, $\overline{n} = 4.7$, $\overline{m} = 4.6$, $N = 16$. [back to photodetector calibration]



Conclusions

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Universal quantum detectors

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Absolute quantum calibration

quantum information theory group

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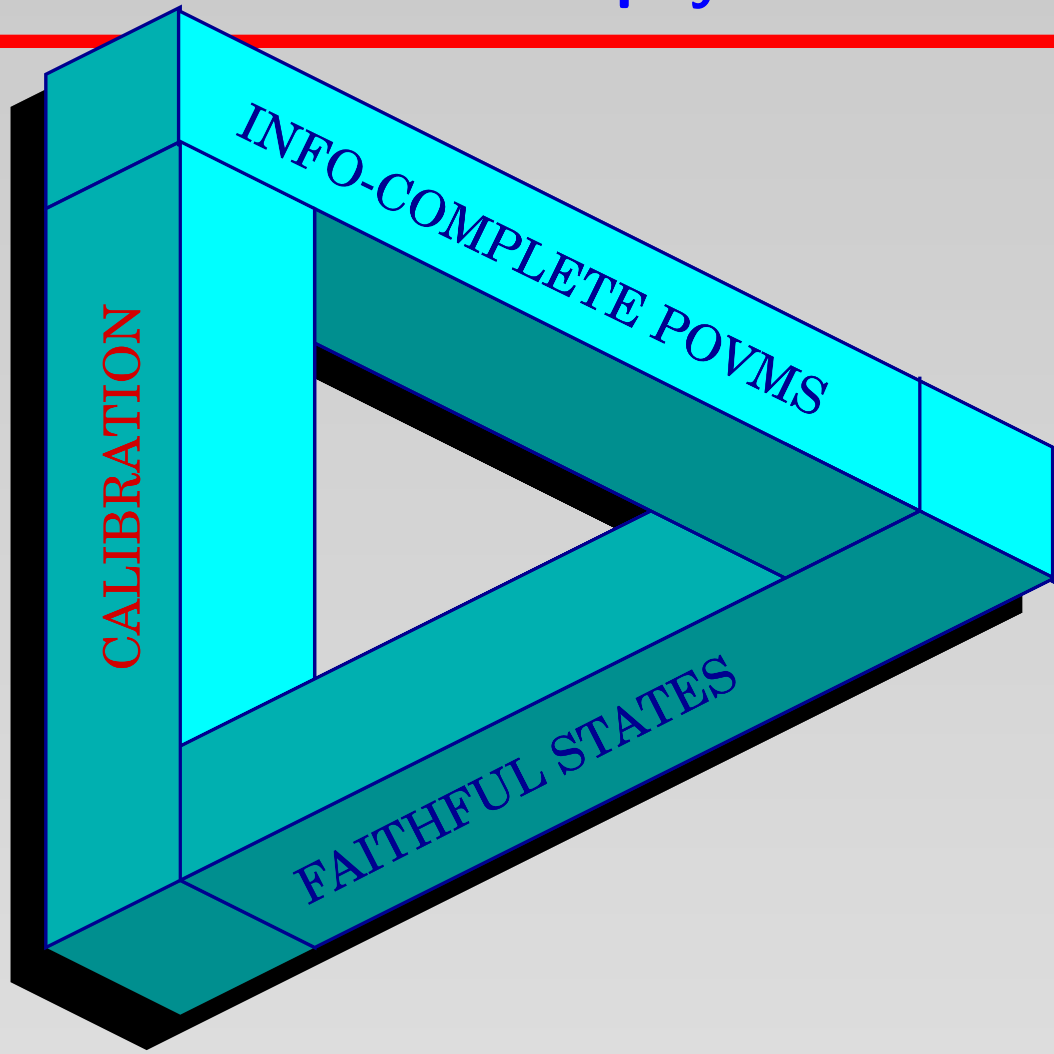
3. The method is robust to detection noise and to mixing of the input state.

Quantum Mechanics: physical axioms?



Informationally complete POVM's = **calibrators**: *"the quantum standards of the International Bureau of Weights and Measures à Paris"* — Chris Fuchs.

Subject Index