Programmability of measurements and channels

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Programmability of measurements and channels

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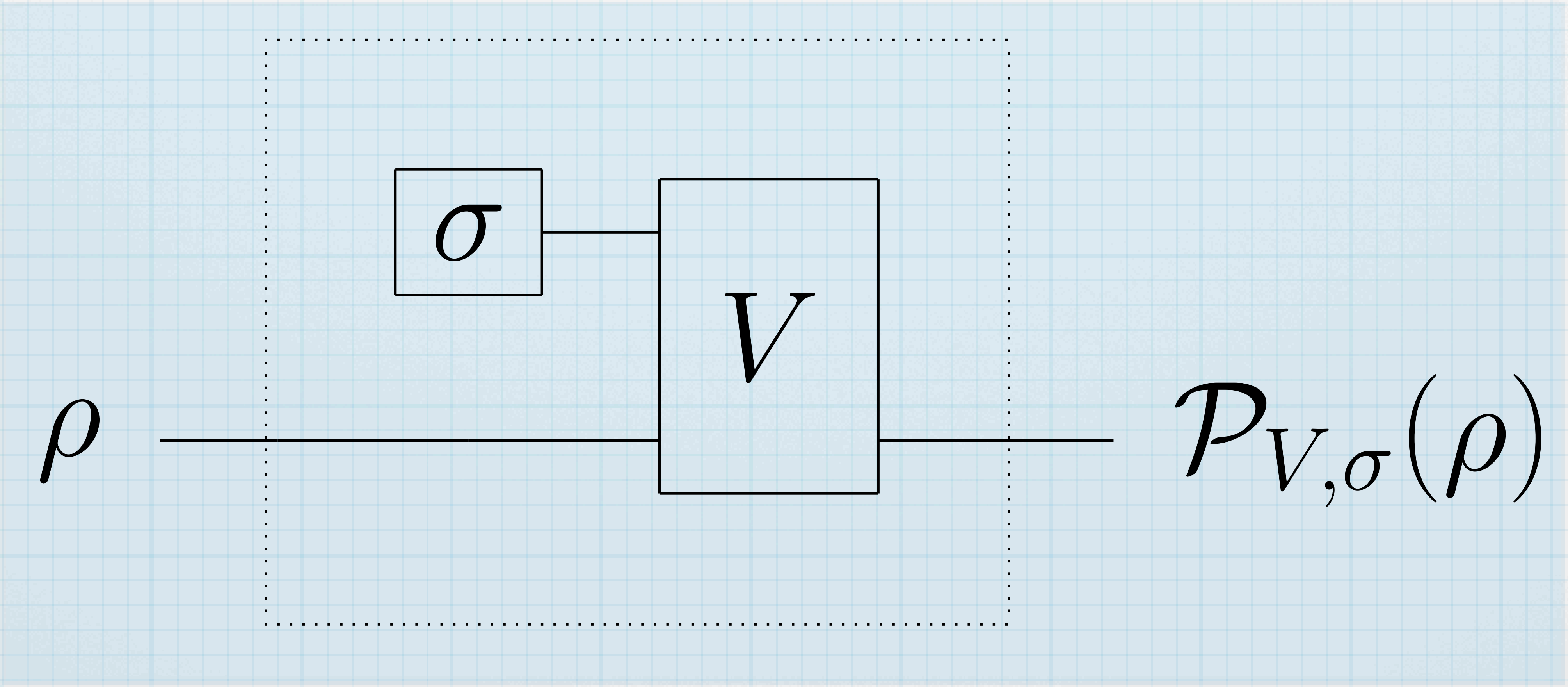
G. M. D'Ariano and P. Perinotti, *Phys. Rev. Lett.* **94** 090401 (2005)
G. M. D'Ariano and P. Perinotti, quant-ph/0509183
G. M. D'Ariano and P. Perinotti, *Phys. Lett.* A **329** 188 (2004)

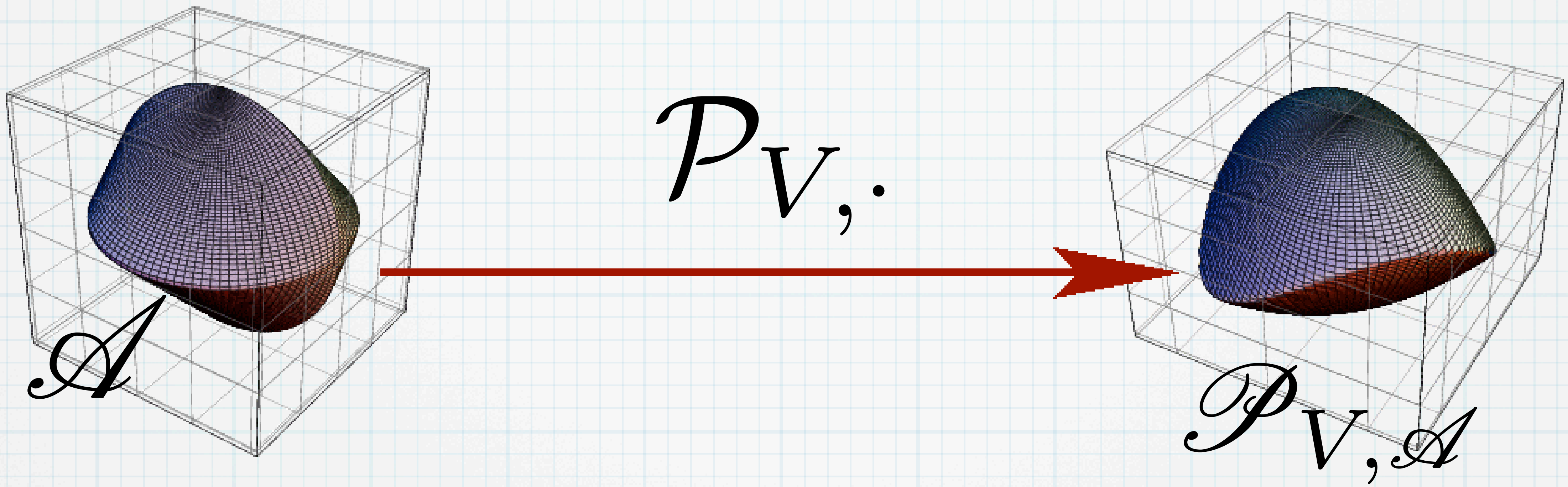
Programmability of channels

Deterministic programming

$$\mathcal{P}_{V,\sigma}(\rho) \doteq \mathrm{Tr}_2[V(\rho \otimes \sigma)V^\dagger]$$



Programmability of channels

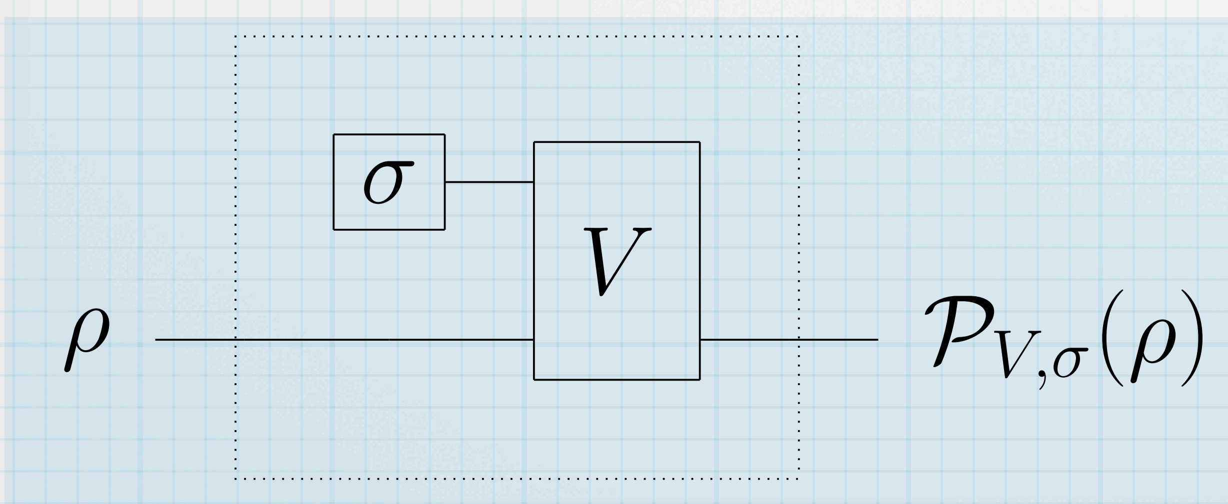


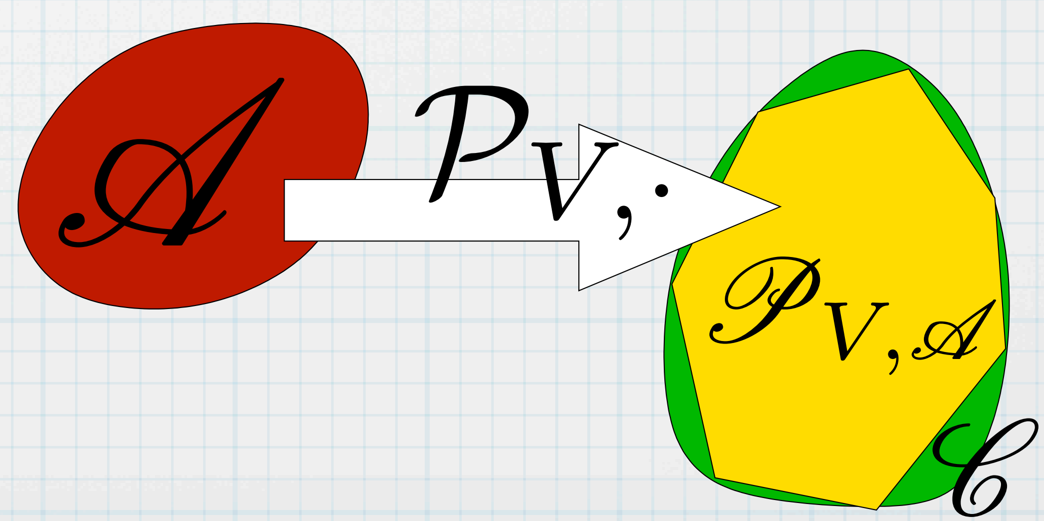
No go theorem (Nielsen-Chuang)

$$\mathcal{P}_{V,\sigma}(\rho) \doteq \mathrm{Tr}_2[V(\rho \otimes \sigma)V^\dagger]$$

It is impossible to program all unitary channels with a single V and a finite-dimensional ancilla

$$\mathscr{P}_V \doteq \mathcal{P}_{V,\mathscr{A}}$$





Programmability of unitaries

Nielsen-Chuang theorem

Suppose distinct (up to a global phase) unitary operators $U_1, \dots U_N$ *are implemented by some programmable quantum gate array. Then the program register is at least N dimensional. Moreover, the corresponding programs* $|\psi_1\rangle, \dots, |\psi_N\rangle$ *are mutually orthogonal.*

Proof: for arbitrary $|\psi\rangle \in \mathsf{H}$

$$V(|\psi\rangle \otimes |\psi_i\rangle) = U_i|\psi\rangle \otimes |\psi'_i\rangle$$

$$V(|\psi\rangle \otimes |\psi_j\rangle) = U_j|\psi\rangle \otimes |\psi'_j\rangle$$

$$\Rightarrow \quad \langle\psi_i|\psi_j\rangle = \langle\psi'_i|\psi'_j\rangle\langle\psi|U_i^\dagger U_j|\psi\rangle$$

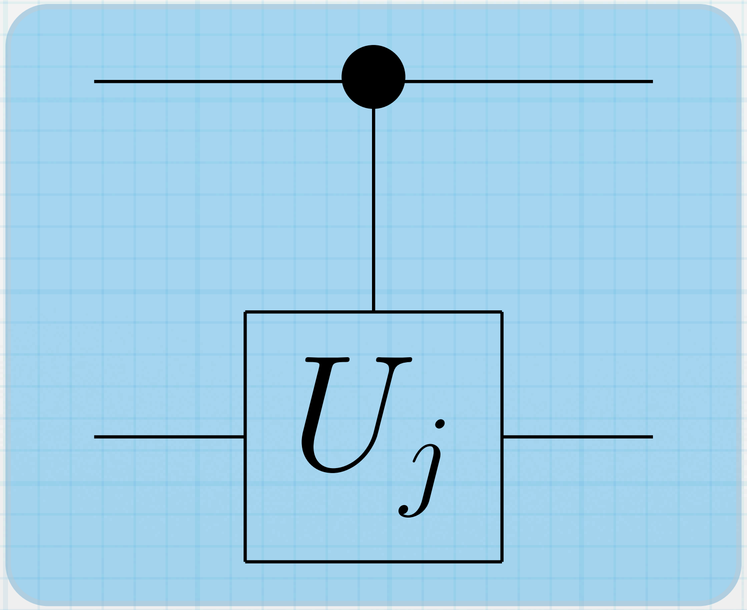
$$\langle\psi'_i|\psi'_j\rangle \neq 0 \ \Rightarrow \frac{\langle\psi_i|\psi_j\rangle}{\langle\psi'_i|\psi'_j\rangle} = \langle\psi|U_i^\dagger U_j|\psi\rangle \qquad \boxed{\Rightarrow \ U_i^\dagger U_j = cI}$$

$$\langle\psi'_i|\psi'_j\rangle = 0 \ \Rightarrow \langle\psi_i|\psi_j\rangle = 0 \ \blacksquare$$

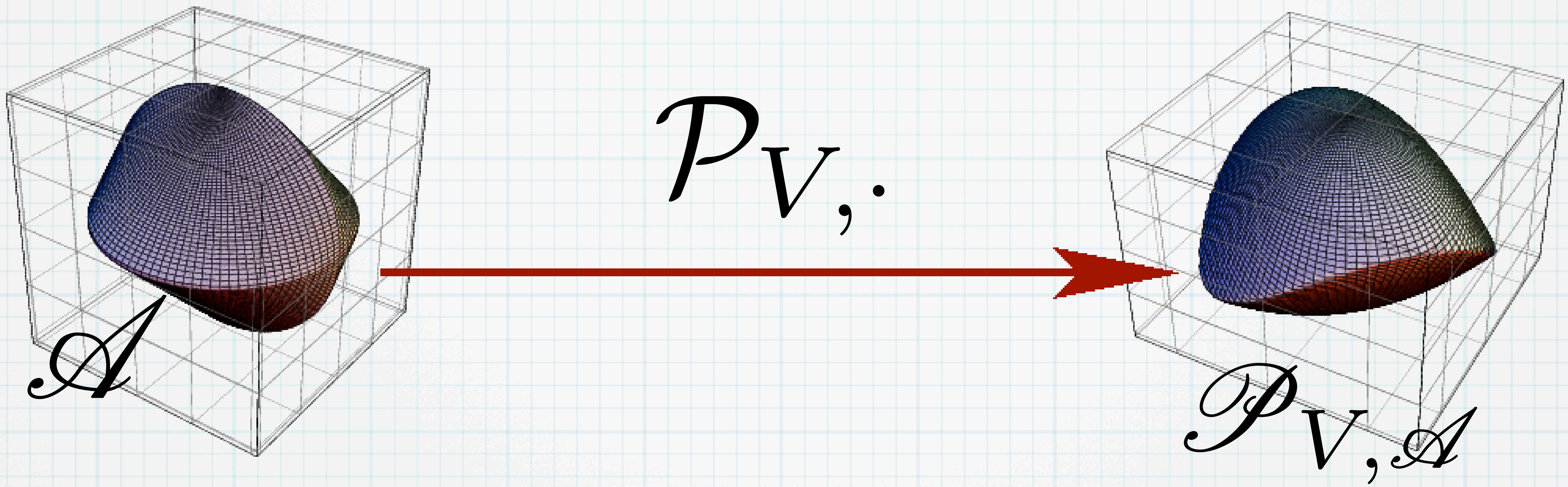
Programmability of unitaries

The joint unitary that programs perfectly the unitary operators $U_1, \ldots U_N$ *is the controlled-U operator (modulo local unitaries)*

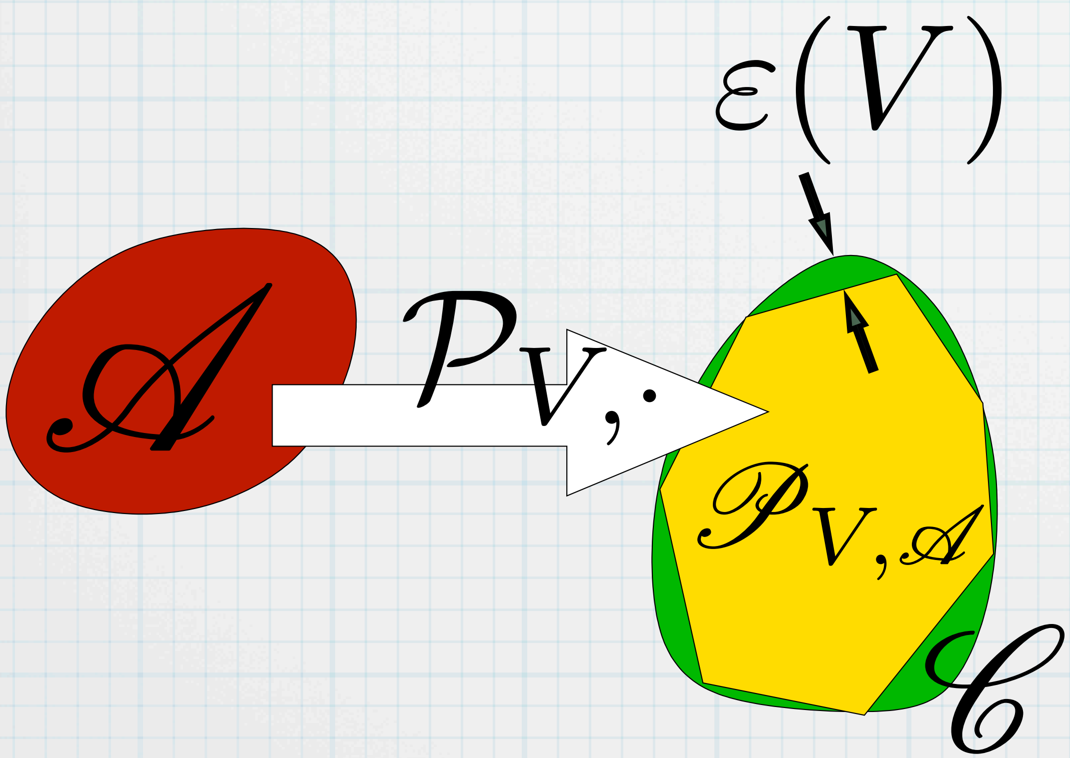
$$V = \sum_j U_j \otimes |\psi_j\rangle\langle\psi_j|$$



The most efficient Unitary



Problem: *The most efficient Unitary*



For given $d = \dim(\mathscr{A})$ find the unitary operators V that are the most efficient in programming channels, namely which minimize the largest distance of each channel $\mathcal{C} \in \mathscr{C}$ from the programmable set $\mathscr{P}_{V,\mathscr{A}}$:

$$\varepsilon(V) \doteq \max_{\mathcal{C}\in\mathscr{C}} \min_{\mathcal{P}\in\mathscr{P}_{V,\mathscr{A}}} \delta(\mathcal{C}, \mathcal{P})$$

The most efficient Unitary

one would like to use $\delta(\mathcal{C},\mathcal{P}) = \|\mathcal{C}-\mathcal{P}\|_{CB}$

instead we use

$$\delta(\mathcal{C},\mathcal{P}) \doteq \sqrt{1-F(\mathcal{C},\mathcal{P})}$$

where $F(\mathcal{C},\mathcal{P})$ is the Raginsky fidelity

$$F(\mathcal{U},\mathcal{P}) = \frac{1}{d^2}\sum_i |\operatorname{Tr}[C_i^\dagger U]|^2 \qquad \mathcal{U} = U\cdot U^\dagger$$

$$F(V) \doteq \min_{U\in\mathsf{U}(\mathsf{H})} F(U,V), \qquad F(U,V) \doteq \max_{\sigma\in\mathscr{A}} F(\mathcal{U},\mathcal{P}_{V,\sigma})$$

The most efficient Unitary

GNS representation

- Bipartite states $|\Psi\rangle\rangle \in \mathsf{H} \otimes \mathsf{K} \iff$ operators $\Psi \in \mathsf{HS}(\mathsf{K}, \mathsf{H})$

$$|\Psi\rangle\rangle = \sum_{nm} \Psi_{nm} |n\rangle \otimes |m\rangle.$$

- Matrix notation (for fixed reference basis in the Hilbert spaces)

$$A \otimes B |C\rangle\rangle = |AC\, B^{\intercal}\rangle\rangle,$$

$$\langle\langle A|B\rangle\rangle \equiv \mathrm{Tr}[A^{\dagger} B].$$

The most efficient Unitary

GNS representation

cyclic vector $|I\rangle\rangle \in \mathsf{H} \otimes \mathsf{K}$

$\Psi \in \mathsf{HS}(\mathsf{K}, \mathsf{H}), \qquad |\Psi\rangle\rangle = (\Psi \otimes I)|I\rangle\rangle$

transposition

$$|\Psi\rangle\rangle = (\Psi \otimes I)|I\rangle\rangle = (I \otimes \Psi^\intercal)|I\rangle\rangle$$

complex conjugation

$$X^* \doteq (X^\intercal)^\dagger$$

$$(|v\rangle\langle v| \otimes I)|I\rangle\rangle = |v\rangle|v^*\rangle$$

The most efficient Unitary

$$V = \sum_k e^{i\theta_k} |\Psi_k\rangle\rangle\langle\langle\Psi_k|;$$

Krauss form

$$\mathcal{P}_{V,\sigma}(\rho) = \sum_{nm} C_{nm}\rho C_{nm}^\dagger, \qquad C_{nm} = \sum_k e^{i\theta_k}\Psi_k|v_n^*\rangle\langle v_m^*|\Psi_k^\dagger\sqrt{\lambda_m}$$

$|v_n\rangle$ denotes the eigenvector of σ corresponding to the eigenvalue λ_n

$$\sum_{nm} |\operatorname{Tr}[C_{nm}^\dagger U]|^2 = \sum_{kh} e^{i(\theta_k-\theta_h)} \operatorname{Tr}[\Psi_k^\dagger U^\dagger \Psi_k \sigma^\intercal \Psi_h^\dagger U \Psi_h]$$
$$= \operatorname{Tr}[\sigma^\intercal S(U,V)^\dagger S(U,V)]$$

$$S(U,V) = \sum_k e^{-i\theta_k}\Psi_k^\dagger U \Psi_k \, .$$

Max over σ

$$F(U,V) = \frac{1}{d^2}\|S(U,V)\|^2.$$

The most efficient Unitary

$$F(U,V) = \frac{1}{d^2}\|S(U,V)\|^2.$$

$$S(U,V) = \mathrm{Tr}_1[(U^\intercal \otimes I)V^*]$$

Changing V by local unitary operators transforms $S(U,V)$ in the following fashion

$$S(U,(W_1 \otimes W_2)V(W_3 \otimes W_4)) = W_2^* S(W_1^\dagger U W_3^\dagger, V)W_4^*$$

The most efficient Unitary d=2

$$V = (W_1 \otimes W_2) \exp[i(\alpha_1 \sigma_1 \otimes {\sigma_1}^{\intercal} + \alpha_2 \sigma_2 \otimes {\sigma_2}^{\intercal} + \alpha_3 \sigma_3 \otimes {\sigma_3}^{\intercal})](W_3 \otimes W_4)$$

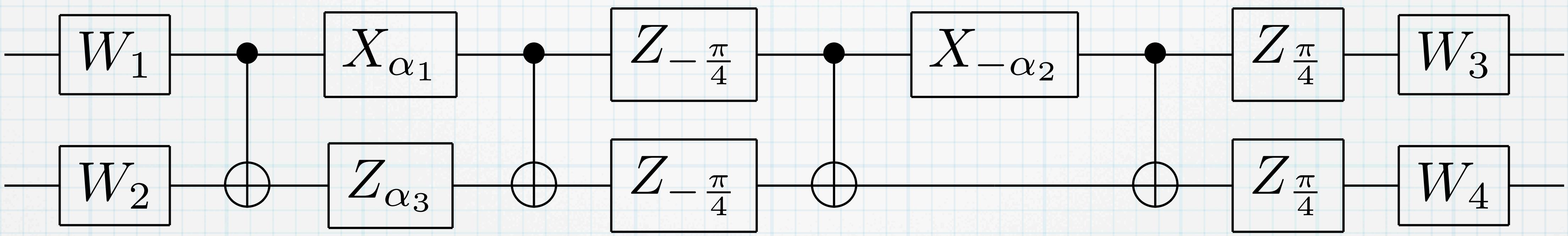


Figure 1. Quantum circuit scheme for the general joint unitary operator V
Here we use the notation $G_\phi = \exp(i\phi\sigma_G)$ with $G = X, Y, Z$.

study only joint unitary operators of the form

$$V = \exp[(i(\alpha_1 \sigma_1 \otimes {\sigma_1}^{\intercal} + \alpha_2 \sigma_2 \otimes {\sigma_2}^{\intercal} + \alpha_3 \sigma_3 \otimes {\sigma_3}^{\intercal})]$$

The most efficient Unitary d=2

$$V = \exp[(i(\alpha_1\sigma_1 \otimes {\sigma_1}^{\mathsf{T}} + \alpha_2\sigma_2 \otimes {\sigma_2}^{\mathsf{T}} + \alpha_3\sigma_3 \otimes {\sigma_3}^{\mathsf{T}})]$$

eigenvectors: $|\Psi_j\rangle\rangle = \frac{1}{\sqrt{2}}|\sigma_j\rangle\rangle$ *(Bell basis)*

$$S(U,V) = \frac{1}{2}\sum_{j=0}^{3} e^{-i\theta_j}\sigma_j U \sigma_j \qquad \theta_0 = \alpha_1 + \alpha_2 + \alpha_3\,, \quad \theta_i = 2\alpha_i - \theta_0$$

Bloch representation:

$$U = n_0 I + i\mathbf{n}\cdot\sigma$$

$$n_k \in \mathbb{R} \text{ and } n_0^2 + |\mathbf{n}|^2 = 1$$

$$S(U,V) = \tilde{n}_0 I + \tilde{\mathbf{n}}\cdot\sigma,$$

$$\tilde{n}_j = t_j n_j, \quad 0 \le j \le 3, t_0 = \frac{1}{2}\sum_{j=0}^{3} e^{-i\theta_j},$$

$$t_j = e^{-i\theta_0} + e^{-i\theta_j} - t_0,\ 1 \le j \le 3, \qquad t_j = |t_j|e^{i\phi_j},\ 0 \le j \le 3,$$

The most efficient Unitary d=2

$$V = \exp[(i(\alpha_1 \sigma_1 \otimes {\sigma_1}^{\intercal} + \alpha_2 \sigma_2 \otimes {\sigma_2}^{\intercal} + \alpha_3 \sigma_3 \otimes {\sigma_3}^{\intercal})]$$

$$\|S(U,V)\|^2 = \mathbf{u} \cdot \mathbf{t} + \sqrt{\mathbf{u} \cdot \mathbf{T} \mathbf{u}}$$

where $\mathbf{u} = (n_0^2, n_1^2, n_2^2, n_3^2)$, $\mathbf{t} = (|t_0|^2, |t_1|^2, |t_2|^2, |t_3|^2)$, and $\mathbf{T}_{ij} = |t_i|^2 |t_j|^2 \sin^2(\phi_i - \phi_j)$. One has the bounds

$$\mathbf{u} \cdot \mathbf{t} + \sqrt{\mathbf{u} \cdot \mathbf{T} \mathbf{u}} \geq \mathbf{u} \cdot \mathbf{t} \geq \min_j |t_j|^2$$

$$F(V) = \frac{1}{d^2} \min_j |t_j|^2.$$

take $\theta_0 = 0, \theta_1 = \pi/2, \theta_2 = \pi, \theta_3 = \pi/2$, corresponding to the eigenvalues $i, 1, -i, 1$ for V. Another solution is $\theta_0 = 0, \theta_1 = -\pi/2, \theta_2 = \pi, \theta_3 = -\pi/2$.

The most efficient Unitary d=2

$$V = \exp[(i(\alpha_1 \sigma_1 \otimes {\sigma_1}^{\mathsf{T}} + \alpha_2 \sigma_2 \otimes {\sigma_2}^{\mathsf{T}} + \alpha_3 \sigma_3 \otimes {\sigma_3}^{\mathsf{T}}))]$$

$$F \doteq \max_{V \in \mathsf{U}(\mathsf{H}^{\otimes 2})} F(V) = \frac{1}{d^2} = \frac{1}{4}$$

the corresponding optimal V has the form

$$V = \exp\left[\pm i \frac{\pi}{4} \left(\sigma_x \otimes \sigma_x \pm \sigma_z \otimes \sigma_z\right)\right]$$

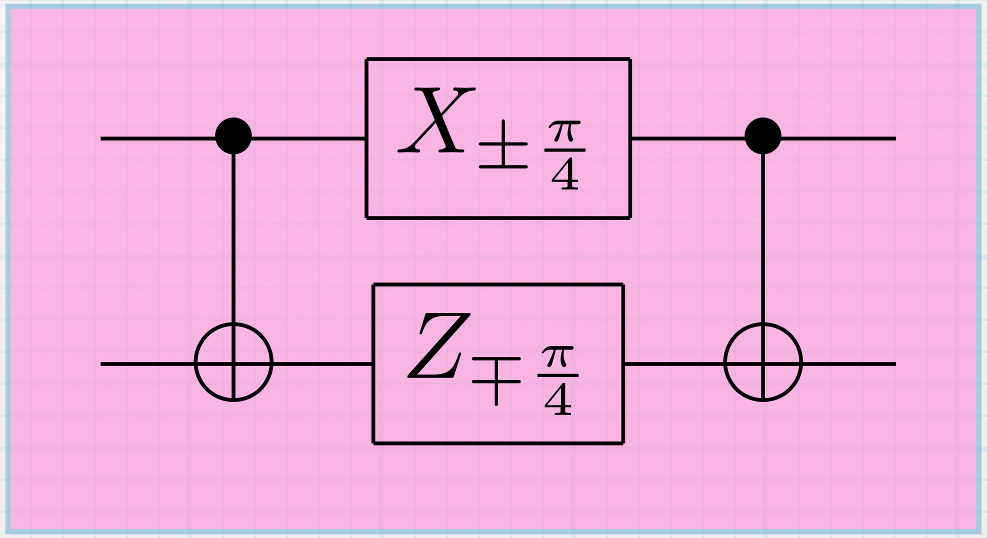


Figure 2. Quantum circuit scheme for the optimal unitary operator V

The most efficient Unitary d=2

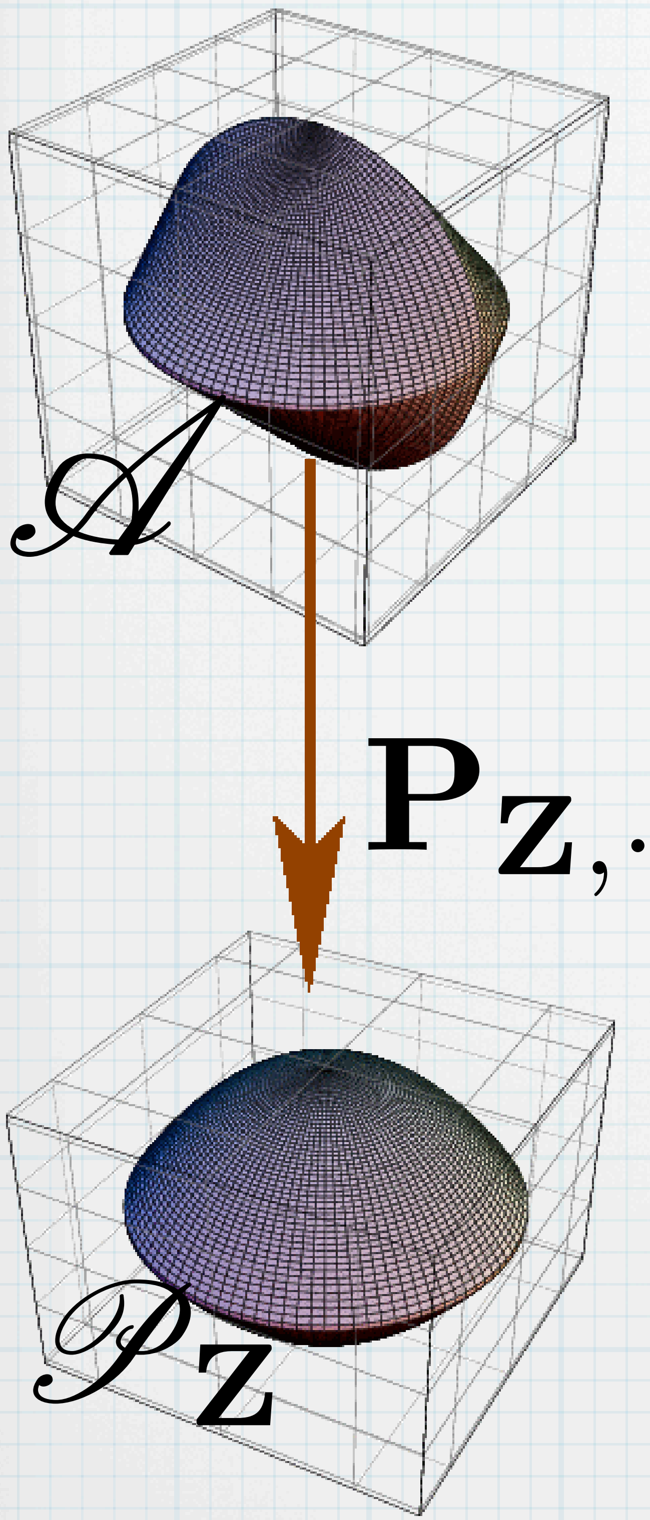
controlled-U

$$V = \sum_{k=1}^{2} V_k \otimes |\psi_k\rangle\langle\psi_k|, \qquad \langle\psi_1|\psi_2\rangle = 0, \quad V_1, V_2 \text{ unitary on } \mathsf{H} \simeq \mathbb{C}^2$$

$$F(U,V) = \frac{1}{4}|\operatorname{Tr}[V_h^\dagger U]|^2 \qquad h = \arg\max_k |\operatorname{Tr}[V_k^\dagger U]|$$

$$F(V) = \min_U F(U,V) = 0.$$

Programmability of POVMs



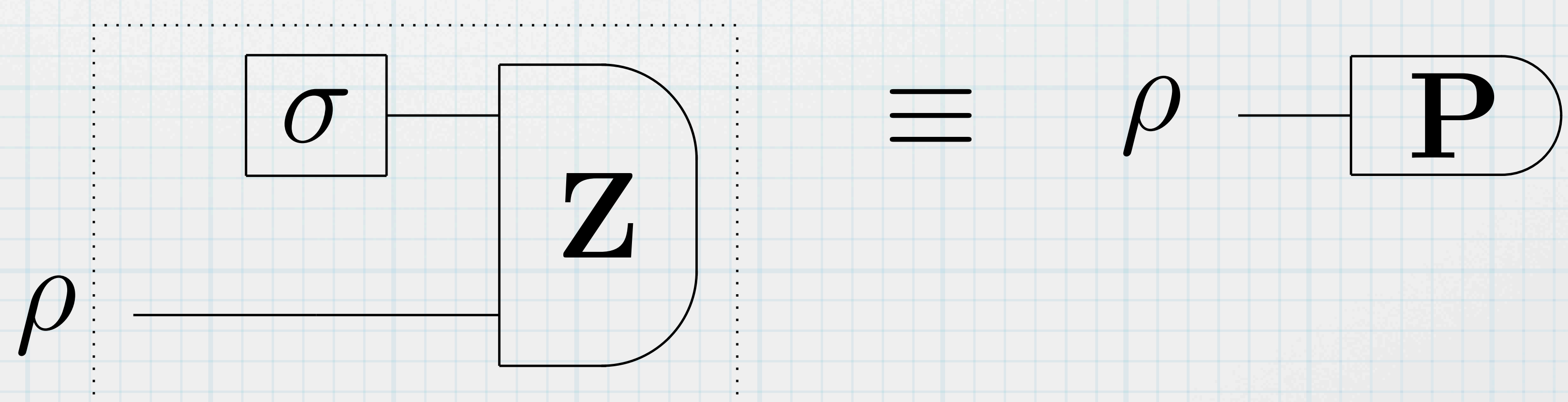
Deterministic

$$\mathbf{P}_{\mathbf{Z},\sigma} \doteq \mathrm{Tr}_2[(I \otimes \sigma)\mathbf{Z}]$$

$$\mathscr{P}_{\mathbf{Z}} \doteq \mathbf{P}_{\mathbf{Z},\mathscr{A}}$$

$$\mathbf{Z} \doteq \{Z_1, Z_2, \ldots, Z_N\}$$

$$\mathbf{P} \doteq \{P_1, P_2, \ldots, P_N\}$$



Programmability of POVMs

Deterministic channel programmability $\Rightarrow$ *Deterministic POVMs programmability*

Deterministic channel programmability $\not\Leftarrow$ *Deterministic POVMs programmability*

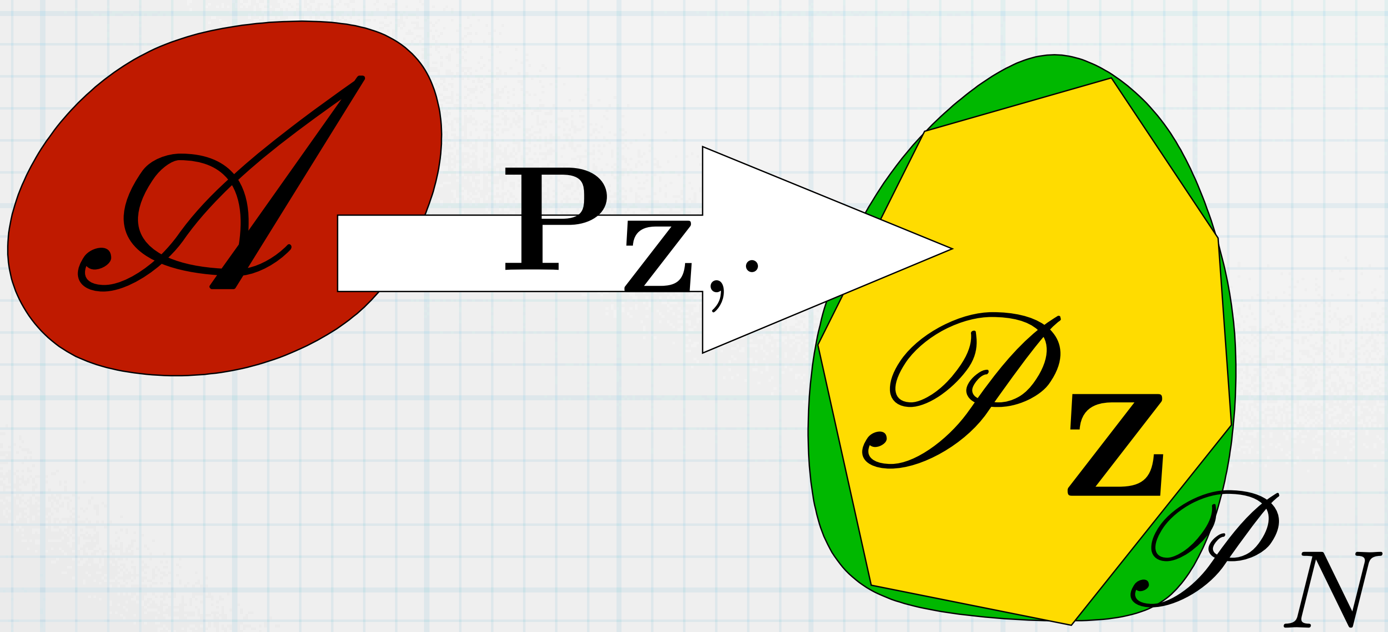
Programmability of POVMs

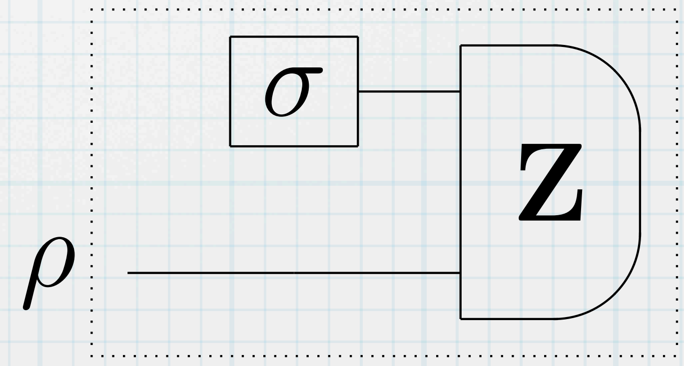
$$\mathbf{P}_{\mathbf{Z},\sigma} \doteq \mathrm{Tr}_2[(I \otimes \sigma)\mathbf{Z}]$$

$$\mathscr{P}_{\mathbf{Z}} \doteq \mathbf{P}_{\mathbf{Z},\mathscr{A}}$$

No go theorem

It is impossible to program all observables with a single joint observable **Z** *and a finite-dimensional ancilla*





Programmability of observables

No go theorem

Suppose M distinct observables $\mathbf{X}_1, \mathbf{X}_2, \ldots, \mathbf{X}_M$ *are implemented by some programmable quantum gate array. Then the program register is at least M dimensional. Moreover, the corresponding programs* $|\psi_1\rangle, \ldots, |\psi_M\rangle$ *are mutually orthogonal.*

Proof: $\boldsymbol{X}_l = \mathrm{Tr}_2[(I \otimes |\psi_l\rangle\langle\psi_l|)\boldsymbol{Z}]$

$$|x_l^{(j)}\rangle\langle x_l^{(j)}| = \mathrm{Tr}_2[(I \otimes |\psi_l\rangle\langle\psi_l|)Z^{(j)}]$$

$$\langle x_l^{(n)}|\langle\psi_l|Z^{(j)}|x_l^{(m)}\rangle|\psi_l\rangle = \delta_{jn}\delta_{jm}$$

$$Z^{(j)}|x_l^{(j)}\rangle|\psi_l\rangle = |x_l^{(j)}\rangle|\psi_l\rangle$$

$$Z^{(i)}Z^{(j)} = \delta_{ij}Z^{(j)} \qquad Z^{(j)}|x_l^{(i)}\rangle|\psi_l\rangle = \delta_{ij}|x_l^{(i)}\rangle|\psi_l\rangle$$

$$\langle\psi_l|\psi_k\rangle\langle x_l^{(j)}|x_k^{(i)}\rangle = 0,\ i \neq j$$

$$\mathbf{X}_l \neq \mathbf{X}_k \ \Rightarrow\ \langle\psi_l|\psi_k\rangle = \delta_{lk} \ \blacksquare$$

Programmability of observables

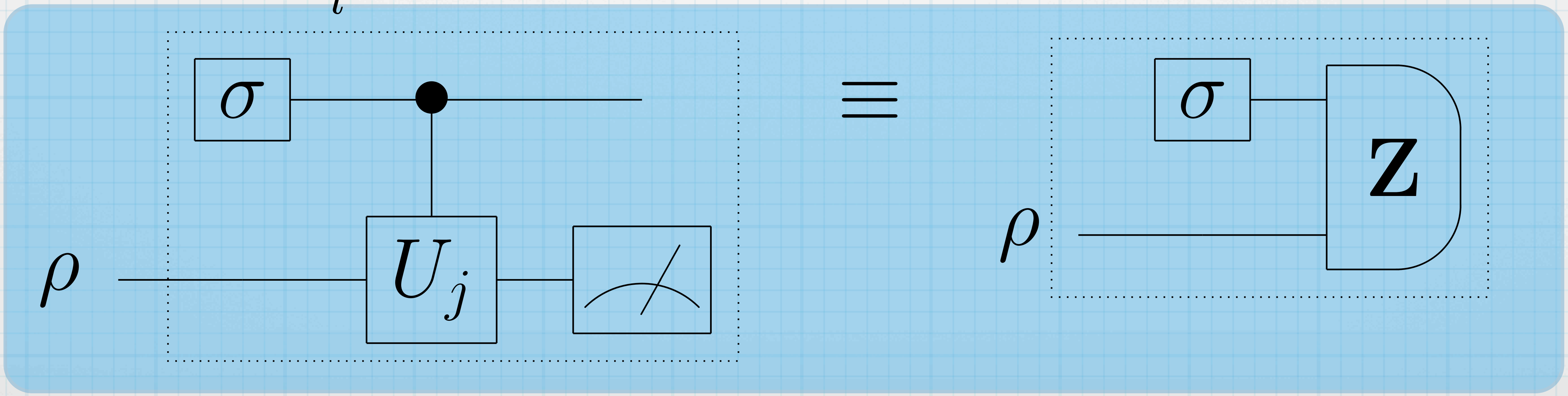
The joint observable that programs perfectly the observables $\mathbf{X}_1, \mathbf{X}_2, \ldots, \mathbf{X}_M$ *is the controlled-O operator*

$$\mathbf{Z} = \sum_l \mathbf{X}_l \otimes |\psi_l\rangle\langle\psi_l|$$

which can be implemented with a fixed local observable and a controlled-U

$$\boxed{\mathbf{Z} = V^\dagger(\mathbf{X} \otimes I)V}$$

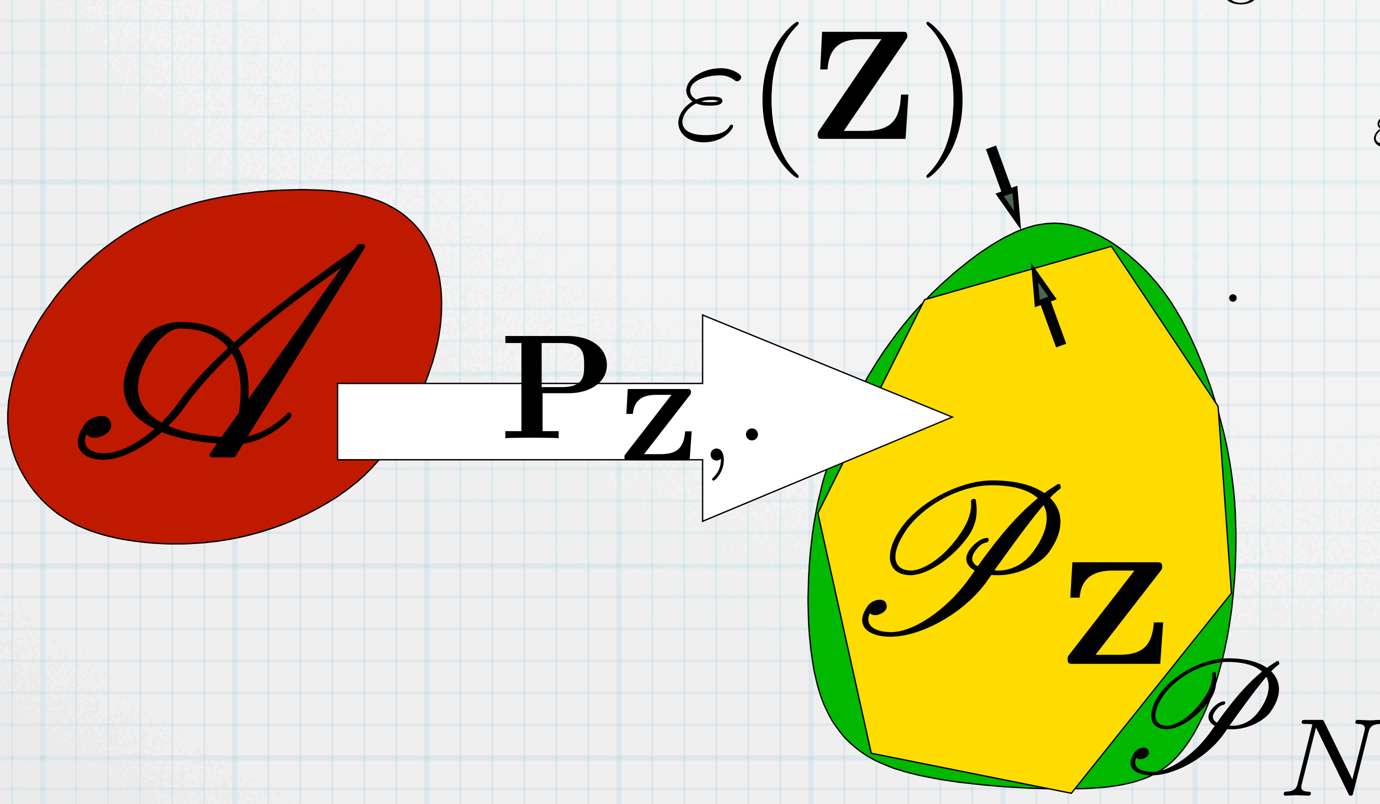
$$V = \sum_l U_l \otimes |\psi_l\rangle\langle\psi_l| \qquad \mathbf{X}_l = U_l^\dagger \mathbf{X} U_l$$



Programmability of observables

$$\mathbf{P}_{\mathbf{Z},\sigma} \doteq \mathrm{Tr}_2[(I \otimes \sigma)\mathbf{Z}]$$

$$\mathscr{P}_{\mathbf{Z}} \doteq \mathbf{P}_{\mathbf{Z},\mathscr{A}}$$



Problem: *The most efficient observable*

For given $d = \dim(\mathscr{A})$ and $N = |\mathbf{Z}| = |\mathbf{P}|$, find the observables $\mathbf{Z}$ that are the most efficient in programming POVM's, namely which minimize the largest distance of each POVM from the programmable set:

$$\varepsilon(\mathbf{Z}) \doteq \max_{\mathbf{Q}\in\mathscr{P}_N} \min_{\mathbf{P}\in\mathscr{P}_{\mathbf{Z},\mathscr{A}}} \delta(\mathbf{P},\mathbf{Q})$$

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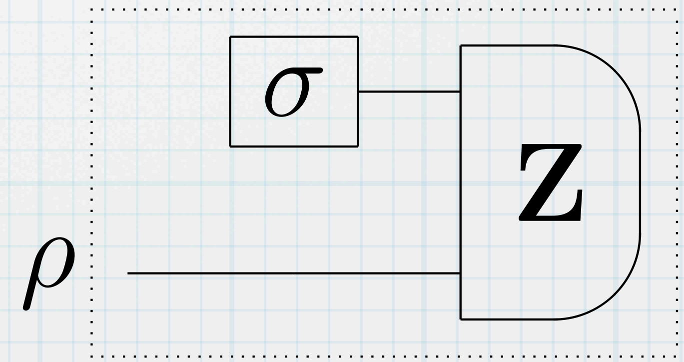
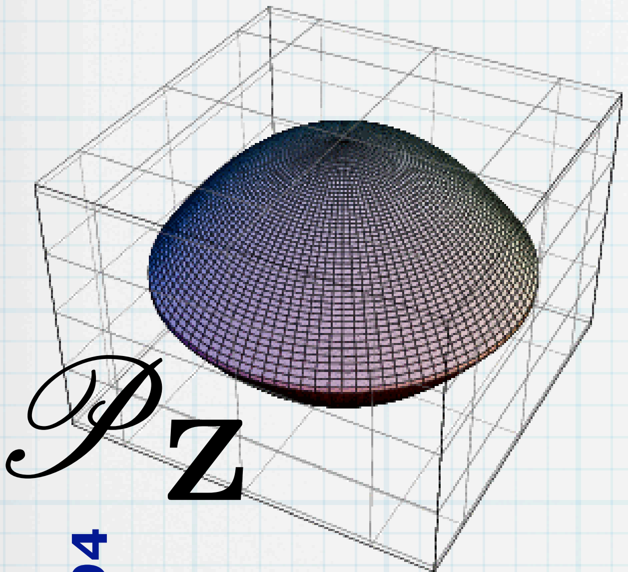


Programmability of observables

programmability with **accuracy** ε^{-1}:



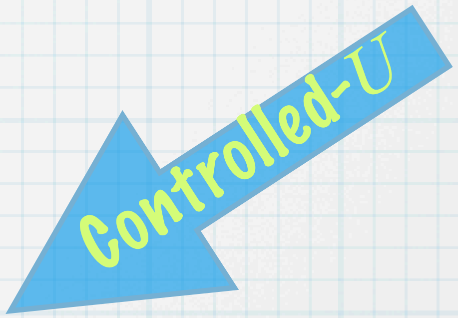
$\mathscr{P}_{\mathbf{Z}}$

$$\varepsilon(\mathbf{Z}) \doteq \max_{\mathbf{Q}\in\mathscr{P}_N} \min_{\mathbf{P}\in\mathscr{P}_{\mathbf{Z}}} \delta(\mathbf{P},\mathbf{Q})$$

$$\delta(\mathbf{P},\mathbf{Q}) = \max_{\rho} \sum_i |\operatorname{Tr}[\rho(P_i - Q_i)]|$$

G. M. D'Ariano, P. Perinotti, Phys. Rev. Lett. **94** 090401 (2005)

Using a joint observable $\mathbf{Z}$ of the form

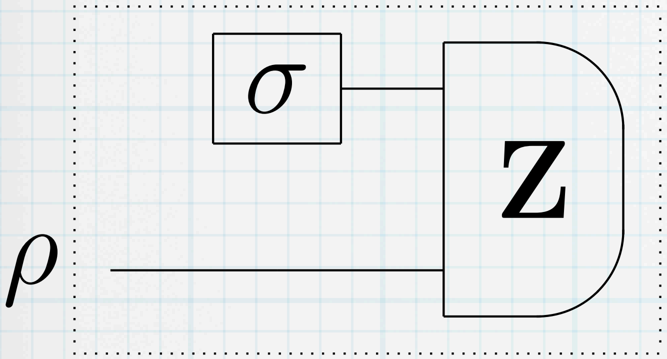


$$Z_i = U^\dagger(|\psi_i\rangle\langle\psi_i| \otimes I_A)U, \qquad U = \sum_{k=1}^{\dim(\mathcal{A})} W_k \otimes |\phi_k\rangle\langle\phi_k|$$

with $\{\psi_i\}$ and $\{\phi_k\}$ orthonormal sets and W_k unitary, we can program observables with accuracy ε^{-1} using an ancilla with **polynomial** growth

$$\dim(\mathcal{A}) \leqslant \kappa(N)\left(\frac{1}{\varepsilon}\right)^{N(N-1)}$$

Programmability of observables d=2



For qubits: *linear* growth!

Program for the observable $\mathbf{P} = \{U_g^{(1/2)} | \pm \frac{1}{2}\rangle\langle \pm \frac{1}{2} | U_g^{(1/2)\dagger}\}$

$$\sigma = U_g^{(j)} |jj\rangle\langle jj| U_g^{(j)\dagger}$$

in dimension $\dim(\mathcal{A}) = 2j + 1$, with joint observable

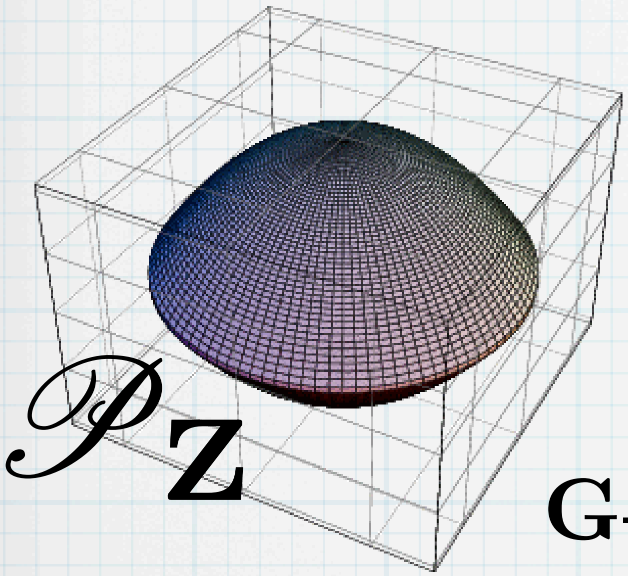
$$\mathbf{Z} = \{\Pi^{(j \pm \frac{1}{2})}\}$$

gives the programmability accuracy

$$\varepsilon(\mathbf{Z}) = \frac{2}{2j+1} \quad \Rightarrow \quad \dim(\mathcal{A}) = 2\varepsilon^{-1}$$

G. M. D'Ariano, P. Perinotti, Phys. Rev. Lett. **94** 090401 (2005)

Exact Programmability of POVMs



Covariant measurements are exactly programmable

$\mathbf{G}$-covariant POVM densities (Holevo theorem)

$$P_g \,\mathrm{d}\, g = U_g \xi U_g^\dagger \,\mathrm{d}\, g, \qquad g \in \mathbf{G}$$

programmable as

$$P_g = \mathrm{Tr}_2[(I \otimes \sigma) F_g], \qquad \xi = V \sigma^\intercal V^\dagger$$

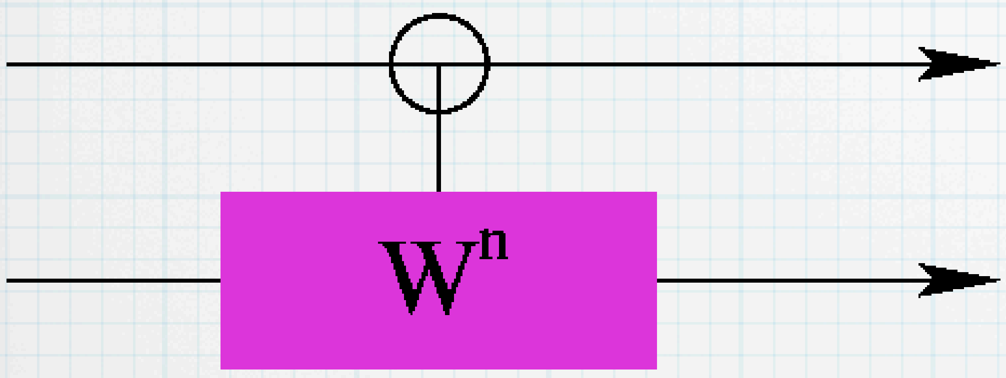
with covariant Bell POVM density

$$F_g = (U_g \otimes I)|V\rangle\!\rangle\langle\!\langle V|(U_g^\dagger \otimes I)$$

G. M. D'Ariano, P. Perinotti, Phys. Lett. A **329** 188 (2004)

Exact Programmability of POVMs

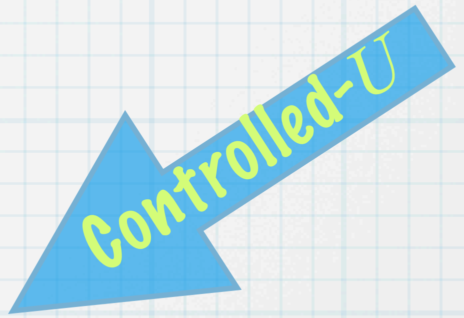
G. M. D'Ariano and P. Perinotti, Phys. Lett A **329** 188-192 (2004)



Unitary operator U connecting the Bell observable with local observables

$$U(|m\rangle \otimes |n\rangle) = \frac{1}{\sqrt{d}} |U_{m,n}\rangle\!\rangle$$

of the controlled-U form



$$U = \sum_n |n\rangle\langle n| \otimes W^n$$

e. g. for projective d-dimensional UIR of the Abelian group $\mathbf{G} = \mathbf{Z}_d \times \mathbf{Z}_d$

$$U_{m,n} = Z^m W^n, \quad Z = \sum_j \omega^j |j\rangle\langle j|, \quad W = \sum_k |k\rangle\langle k \oplus 1|, \quad \omega = e^{\frac{2\pi i}{d}}.$$

Conclusions

- Programmable channels:
  - Nielsen-Chuang theorem revisited
  - Exact programming for finite set of unitaries: controlled-U
  - Optimal programming in 2x2 dimensions: two controlled-NOT
- Programmable POVMs:
  - No go theorem
  - Exact programming for finite set of observables: controlled-O
  - controlled-O: polynomial complexity programming
  - for qubits: linear complexity programming