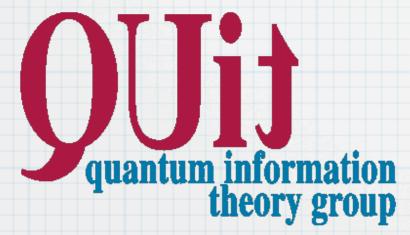
# Broadcasting quantum information

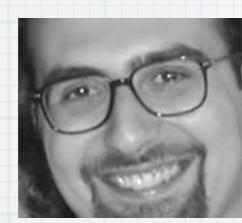
Giacomo Mauro D'Ariano

Dipartimento di Fisica "A. Volta", Università di Pavia



18-19 November 2005 Royal Danish Academy of Sciences and Letters, Copenhagen

# Superbroadcasting



Buscemi



60

Macchiavello



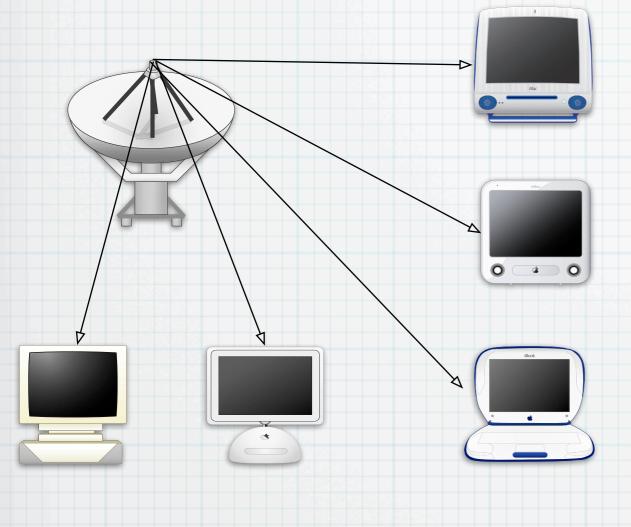
Perinotti

G. M. D'Ariano, C. Macchiavello, and P. Perinotti,

Superbroadcasting of mixed states, Phys. Rev. Lett. 95 060503 (2005)

**F. Buscemi, G. M. D'Ariano, C. Macchiavello, and P. Perinotti,** (in preparation)

# Broadcasting



"Information" is by its nature **broadcastable**.

# What about when information is *quantum*?

- Distributed quantum computation
- Quantum secret sharing
- Quantum game-theoretical contexts...

Broadcasting quantum information can be done only in a limited fashion

# No cloning theorem

 $N \text{ inputs} \Rightarrow M \text{ outputs}$ 

$$\underbrace{|\psi\rangle \otimes \ldots \otimes |\psi\rangle}_{N} \Longrightarrow \underbrace{|\psi\rangle \otimes |\psi\rangle \otimes \ldots \otimes |\psi\rangle}_{M}, \quad \forall |\psi\rangle \in \mathsf{H}$$

For *M*>*N* the transformation cannot be achieved isometrically, whence it cannot occur with unit probability.

$$|E\rangle \otimes \underbrace{|\psi\rangle \otimes \ldots \otimes |\psi\rangle}_{N} \otimes |\omega_{1}\rangle \otimes \ldots \otimes |\omega_{M-N}\rangle \Longrightarrow |E_{\psi}\rangle \otimes \underbrace{|\psi\rangle \otimes |\psi\rangle \otimes \ldots \otimes |\psi\rangle}_{M}$$

 $|\langle \varphi | \psi \rangle|^{N} \Longrightarrow |\langle E_{\varphi} | E_{\psi} \rangle ||\langle \varphi | \psi \rangle|^{M}$ 

# Cloning/Broadcasting

 $N \text{ inputs} \Rightarrow M \text{ outputs}$ 

 $R_{out} = \rho \otimes \rho \otimes \ldots \otimes \rho \qquad "cloning"$  $\mathrm{Tr}_{123...M-1}[R_{out}] = \mathrm{Tr}_{23...M}[R_{out}] = \rho \qquad "broadcasting"$ 

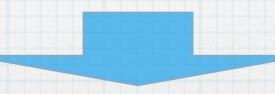
• For pure states ideal broadcasting coincides with the *quantum cloning*.

• For mixed states there are infinitely many joint states that correspond to the same local state.

# No-broadcasting

• For mixed input states the *no-cloning theorem* is not logically sufficient to forbid ideal broadcasting

 The *impossibility of ideal broadcasting* has been proved in the case of <u>one input</u> copy and <u>two output</u> copies for *non mutually commuting density operators* [H. Barnum, C. M. Caves, C. A. Fuchs, R. Jozsa, and B. Schumacher, Phys. Rev. Lett. **76** 2818 (1996)]

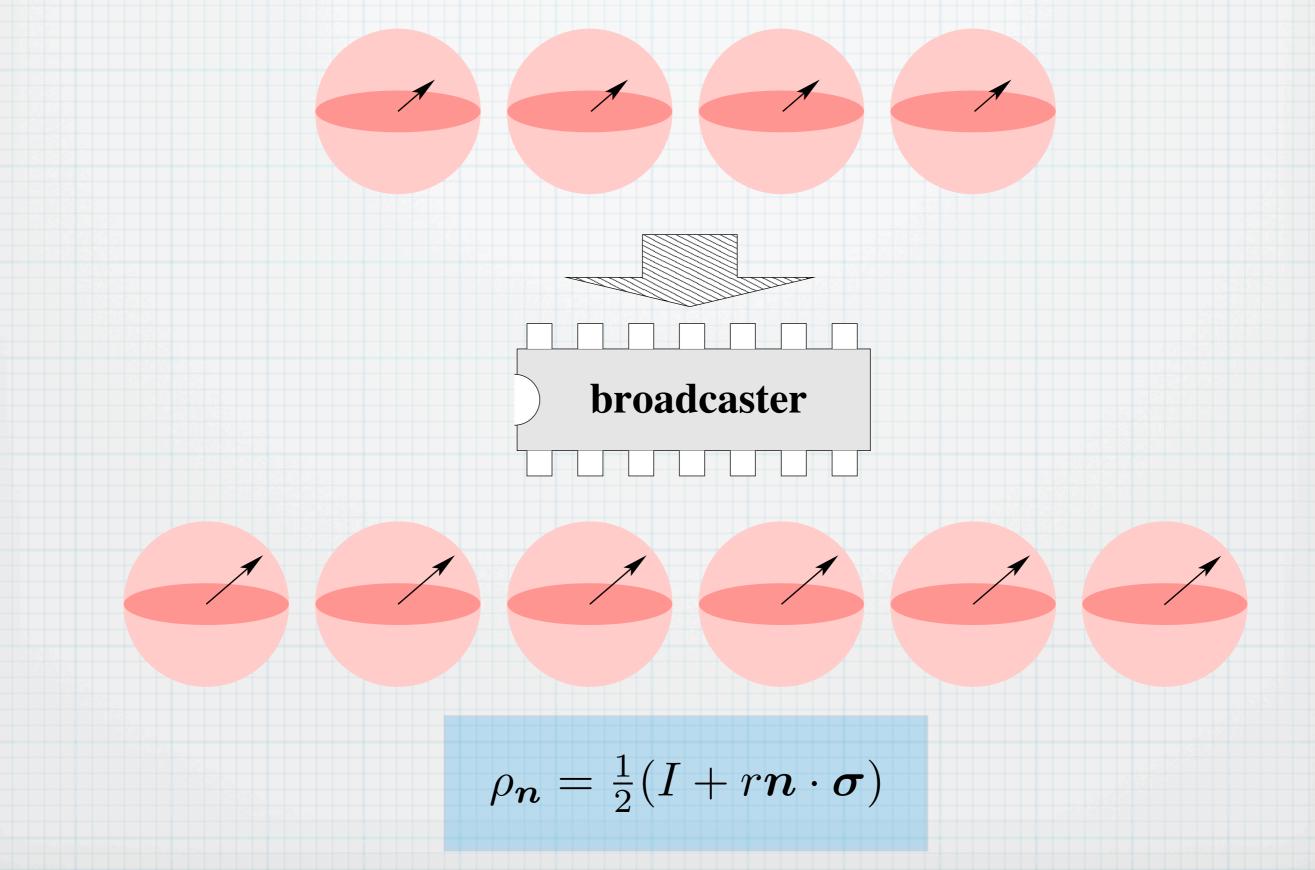


Is this a generalization of the no-cloning theorem to mixed states?

# Superbroadcasting

#### The answer is no!

- The no broadcasting theorem does not generalize to multiple input copies!
- For sufficiently many input copies it is even possible to purify the state while broadcasting!
- broadcasting + purification: "superbroadcasting".



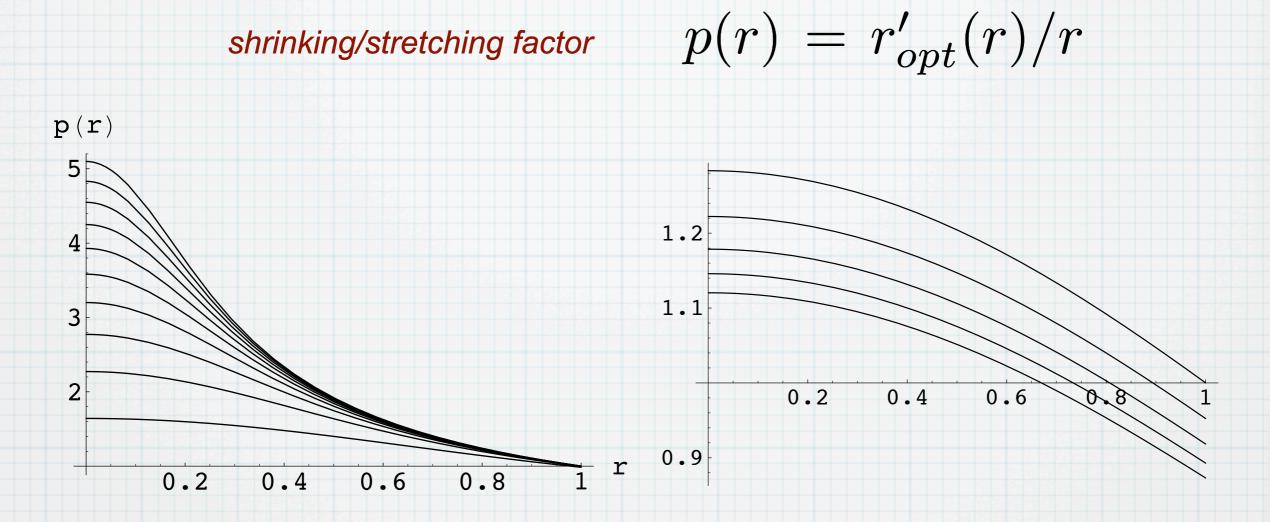
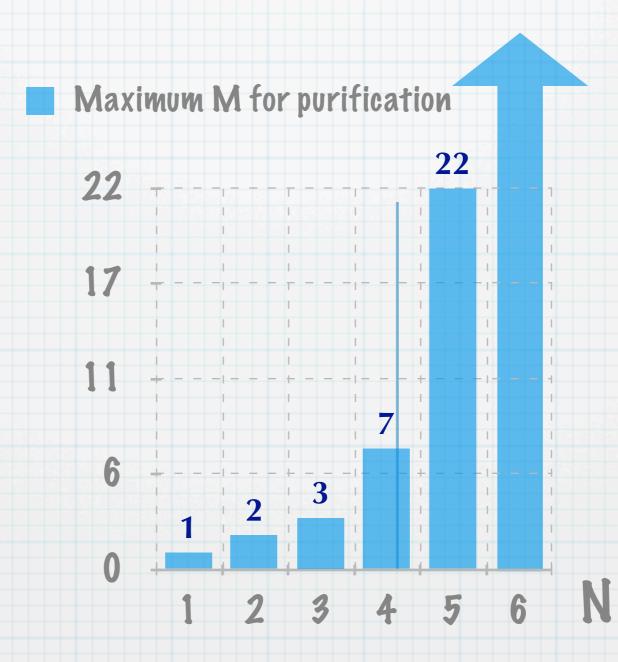


FIG. 2: The stretching factor p(r) versus r. On the left: for M = N + 1 and N = 10, 20, 30, 40, 50, 60, 70, 80, 90, 100(from the bottom to the top. On the right: for N = 5 and  $5 \le M \le 9$  (from the top to the bottom).



shrinking/stretching factor  $p(r) = r'_{opt}(r)/r$ 

 $r_st(N,M)$  maximum purity for purification

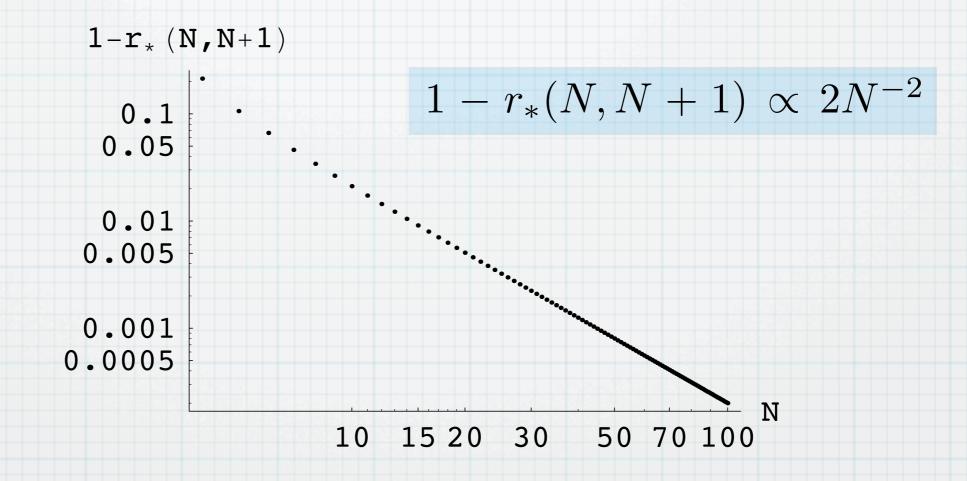


FIG. 3: Logarithmic plot of  $1 - r_*(N, N + 1)$  versus N.  $r_*(N, M)$  denotes the maximum purity for which one can have superbroadcasting from N to M copies.

- For pure states the optimal superbroadcasting map is the same as the optimal universal cloning [R. F. Werner, Phys. Rev. A 58 1827 (1998)].
- For M<N it corresponds to the optimal purification map [ J. I. Cirac, A. K. Ekert, and C. Macchiavello, Phys. Rev. Lett. 82 4344 (1999)].
- Therefore, the superbroadcasting map generalizes and *interpolates optimal purification and optimal cloning*.



#### GNS representation

cyclic vector  $|I\rangle\rangle \in \mathsf{H} \otimes \mathsf{H}$ 

 $\Psi \in \mathsf{HS}(\mathsf{K},\mathsf{H}), \qquad |\Psi\rangle\!\rangle = (\Psi \otimes I)|I\rangle\!\rangle$ 

transposition

$$\begin{split} |\Psi\rangle\rangle &= (\Psi\otimes I)|I\rangle\rangle = (I\otimes\Psi^{\intercal})|I\rangle\rangle\\ complex conjugation\\ X^* &\doteq (X^{\intercal})^{\dagger}\\ (|v\rangle\langle v|\otimes I)|I\rangle\rangle &= |v\rangle|v^*\rangle \end{split}$$

$$|\Psi\rangle\rangle = \sum \Psi_{nm} |n\rangle \otimes |m\rangle.$$

nm

 $(A \otimes B)|C\rangle\rangle = |AC B^{\intercal}\rangle\rangle$  $\langle\langle A|B\rangle\rangle \equiv \operatorname{Tr}[A^{\dagger}B].$ 



# Choi-Jamiolkowski

Choi-Jamiołkowski correspondence

$$\mathcal{M}(U_g \rho U_g^{\dagger}) = W_g \mathcal{M}(\rho) W_g^{\dagger} \Leftrightarrow [W_g \otimes U_g^{*}, R_{\mathcal{M}}] = 0$$
$$\mathcal{M}(T_h \rho T_h^{\dagger}) = \mathcal{M}(\rho) \Leftrightarrow [I \otimes T_h^{*}, R_{\mathcal{M}}] = 0$$
$$V_k \mathcal{M}(\rho) V_k^{\dagger} = \mathcal{M}(\rho) \Leftrightarrow [V_k \otimes I, R_{\mathcal{M}}] = 0$$

# Conjugation/covariance

"conjugation":  $CUC^{\dagger} = U^{*}$   $C \equiv (i\sigma_{y})^{\otimes N}$   $\tilde{Q} \doteq CQ^{\intercal}C^{\dagger}$   $S_{\mathscr{B}} \doteq (I_{out} \otimes C)R_{\mathscr{B}}(I_{out} \otimes C^{\dagger})$  $\mathscr{B}(Q) = \operatorname{Tr}_{in}[(I_{out} \otimes \tilde{Q})S_{\mathscr{B}}]$ 

Completely positive trace-preserving map from states of N qubits to states of M qubits that is invariant under permutations of input copies and of output copies and unitarily covariant

 $\Pi_{\sigma}\mathscr{B}(\Pi_{\tau}\rho\Pi_{\tau}^{\dagger})\Pi_{\sigma}^{\dagger} = \mathscr{B}(\rho) \qquad \mathscr{B}(U^{\otimes N}\rho^{\otimes N}U^{\dagger\otimes N}) = U^{\otimes M}\mathscr{B}(\rho^{\otimes N})U^{\dagger\otimes M}$  $[\Pi_{\sigma}\otimes\Pi_{\tau}, S_{\mathscr{B}}] = 0 \qquad \qquad [U^{\otimes (M+N)}, S_{\mathscr{B}}] = 0$ 

# Schur-Weyl duality

We exploit the Schur-Weyl duality

$$[\mathbb{SU}(d)^{\otimes N}, \mathbb{P}_N] = 0 \qquad U_g^{(\nu)}$$
$$\mathcal{H}^{\otimes N} \equiv \bigoplus_{\nu} \mathcal{H}_{\nu} \otimes \mathcal{H}_{d_{\nu}} \qquad \Pi_{\sigma}^{(d_{\nu})}$$

H<sub>i</sub> representation space  $\mathsf{H}^{\otimes L} = \bigoplus_{j=\langle\!\langle L/2\rangle\!\rangle}^{L/2} \mathsf{H}_j \otimes \mathbb{C}^{d_j}$  $\mathbb{C}^{d_j}$  multiplicity space  $d_j = \frac{2j+1}{L/2+j+1} {L \choose L/2+j}$ 

operators invariant under  $U_g^{\otimes L}$ :  $\oplus_{j=\langle\langle L/2 \rangle\rangle}^{L/2} I_j \otimes W^{(j)}$ 

 $\mathbb{SU}(2)$ 

operators invariant under  $\mathbb{P}_L$ :  $\oplus_{j=\langle\langle L/2 \rangle\rangle}^{L/2} Z_j \otimes I_{d_j}$ 

### Input states

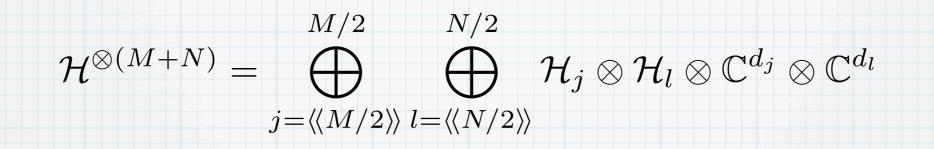
 $\tilde{\rho}^{\otimes N} = (r_+ r_-)^{N/2} \bigoplus_{l=\langle\langle N/2\rangle\rangle}^{N/2} \sum_{n=-l}^l \left(\frac{r_-}{r_+}\right)^n |ln\rangle\langle ln| \otimes I_{d_l}$ 

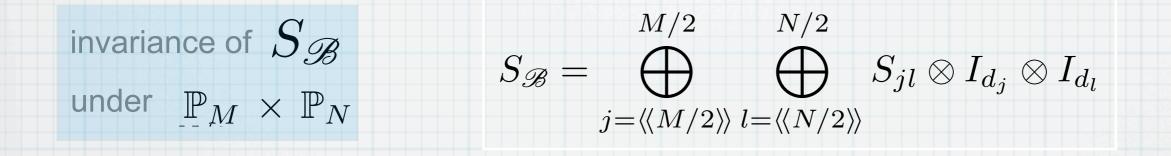
(Schur-Weyl duality)

## $\rho = \frac{1}{2}(I + r\mathbf{k} \cdot \boldsymbol{\sigma}) \qquad r_{\pm} \doteq \frac{1}{2}(1 \pm r)$

[J. I. Cirac, A. K. Ekert, and C. Macchiavello, Phys. Rev. Lett. 82 4344 (1999)]

# Maps characterization





 $S_{\mathscr{B}} = \bigoplus^{M/2} \bigoplus^{N/2} \bigoplus^{j+l} s_{j,l,J} P_J^{(j,l)} \otimes I_{d_j} \otimes I_{d_l}$ invariance of  $S_{\mathscr{B}}$ under  $U_q^{\otimes (N+M)}$  $j = \langle \langle M/2 \rangle \rangle l = \langle \langle N/2 \rangle \rangle J = |j-l|$ 

### Convex structure

$$S_{\mathscr{B}} = \bigoplus_{j=\langle\langle M/2\rangle\rangle}^{M/2} \bigoplus_{l=\langle\langle N/2\rangle\rangle}^{N/2} \bigoplus_{J=|j-l|}^{j+l} s_{j,l,J} P_J^{(j,l)} \otimes I_{d_j} \otimes I_{d_l}$$

 $P_J^{(j,l)}$  orthogonal projector over the irreducible representation *J* coming from the couple *j*,*l* 

 $s_{j,l,J}$  positive coefficients

trace-preserving condition:

$$\sum_{j=\langle\!\langle M/2\rangle\!\rangle}^{M/2} \sum_{J=|j-l|}^{j+l} d_j s_{j,l,J} \frac{2J+1}{2l+1} = 1, \quad \forall \langle\!\langle N/2\rangle\!\rangle \le l \le \frac{N}{2}$$

broadcasting maps make a convex set, with the extreme points classified by the functions  $\varphi$  and  $\Phi$  corresponding to a given choice  $j = \varphi_l$ ,  $J = \Phi_l$ ,  $\langle\langle M/2 \rangle\rangle \leq \varphi_l \leq M/2$ ,  $|\varphi_l - l| \leq \Phi_l \leq \varphi_l + l$ 

# Extremal maps

extremal broadcasting maps:

$$S_{\mathscr{B}}^{(\varphi,\Phi)} = \bigoplus_{l=\langle\langle N/2\rangle\rangle}^{N/2} \frac{2l+1}{2\Phi_l+1} \frac{1}{d_{\varphi_l}} P_{\Phi_l}^{(\varphi_l,l)} \otimes I_{d_{\varphi_l}} \otimes I_{d_l}$$

$$\mathscr{B}_{\varphi,\Phi}(\rho^{\otimes N}) = (r_{+}r_{-})^{N/2} \bigoplus_{l=\langle\langle N/2\rangle\rangle}^{N/2} \frac{2l+1}{2\Phi_{l}+1} \frac{d_{l}}{d_{\varphi_{l}}}$$

$$\times \sum_{n=-l}^{l} \left(\frac{r_{-}}{r_{+}}\right)^{n} \operatorname{Tr}_{l}\left[\left(I_{\varphi_{l}} \otimes |ln\rangle\langle ln|\right)P_{\Phi_{l}}^{(\varphi_{l},l)}\right] \otimes I_{d_{\varphi_{l}}}$$

# Extremal maps

The output state can be written

$$\mathscr{B}_{\varphi,\Phi}(\rho^{\otimes N}) = (r_{+}r_{-})^{N/2} \bigoplus_{l=\langle\langle N/2\rangle\rangle}^{N/2} \frac{2l+1}{2\Phi_{l}+1} \frac{d_{l}}{d_{\varphi_{l}}}$$

$$\times \sum_{n=-l}^{l} \left(\frac{r_{-}}{r_{+}}\right)^{n} \sum_{m=-\varphi_{l}}^{\varphi_{l}} \langle \Phi_{l}m + n | \varphi_{l}m, ln \rangle^{2} | \varphi_{l}m \rangle \langle \varphi_{l}m | \otimes I_{d_{\varphi_{l}}}$$

We are now interested in the single-site output

Let's focus attention on this term

# Single-site output

Change from Wedderburn to qubit representations

$$|jm\rangle \otimes |1\rangle = |jm\rangle \otimes |\Psi_{-}\rangle^{\otimes \frac{M}{2}-j}$$

$$\mathrm{Tr}_{j-\frac{1}{2}}[|jm\rangle\langle jm|] = \frac{1}{2}I + \frac{m}{2j}\boldsymbol{k}\cdot\boldsymbol{\sigma}$$

$$|jm\rangle\langle jm|\otimes I_{d_j} = \frac{d_j}{M!}\sum_{l\in\mathbb{P}_M}\pi_l X_j\pi_l^{\dagger}$$
$$X_j = |jm\rangle\langle jm|\otimes|1\rangle\langle 1|$$

$$\begin{split} \rho_{(\varphi,\Phi)}'(r) &= (r_+ r_-)^{N/2} \sum_{l=\langle\langle N/2\rangle\rangle}^{N/2} \frac{2l+1}{2\Phi(l)+1} d_l \sum_{m=-\varphi(l)}^{\varphi(l)} \\ &\times \sum_{n=-l}^l \left(\frac{r_-}{r_+}\right)^n \langle \Phi(l)m + n | \varphi(l)m, ln \rangle^2 \frac{1}{2} \left(I + \frac{2m}{M} \mathbf{k} \cdot \boldsymbol{\sigma}\right) \end{split}$$

# Derivation

The single-site output state  $\rho' = \operatorname{Tr}_{M-1}[\mathscr{B}(\rho^{\otimes N})]$ 

commutes with  $\sigma_z$ 

As a figure of merit we consider

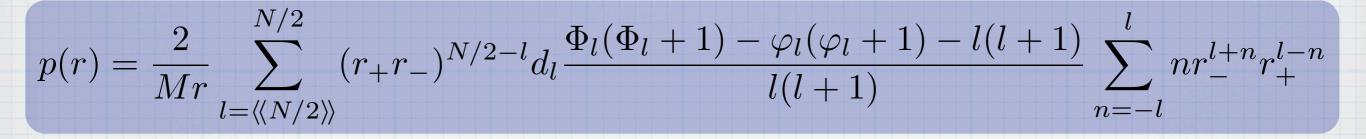
$$p(r) \doteq \frac{r'}{r} = \frac{1}{r} \operatorname{Tr}[\sigma_z \otimes I^{\otimes (M-1)} \mathscr{B}(\rho^{\otimes N})]$$

Using permutation invariance it turns out that

$$p(r) = \frac{2}{Mr} \operatorname{Tr}[J_z^{\text{tot}} \mathscr{B}(\rho^{\otimes N})]$$



For extremal maps we have



Since  $\sum_{n=-l}^{l} nr_{-}^{l+n}r_{+}^{l-n} \le 0$   $\Phi_l, \varphi_l$  must minimize

$$\Phi_l(\Phi_l+1) - \varphi_l(\varphi_l+1) - l(l+1)$$

# Scaling factor

The solution is

$$\varphi_l = \frac{M}{2} \,, \ \Phi_l = \frac{M}{2} - l$$

#### corresponding to

$$p(r) = -\frac{M+2}{Mr} \sum_{l=\langle\langle N/2\rangle\rangle}^{N/2} (r_{+}r_{-})^{N/2-l} \frac{d_{l}}{l+1} \sum_{n=-l}^{l} nr_{-}^{l+n}r_{+}^{l-n}$$



### Violation of data-processing theorem?

- Superbroadcasting doesn't mean more available information about the original input state.
- This is due to detrimental correlations between the broadcast copies, which does not allow to exploit their statistics.
- From the *point of view of each single user* our broadcasting protocol is a purification in all respects (for sufficiently mixed states). The process transfers noise from the local states to the correlations between them.

### Violation of data-processing theorem?

G. M. D'Ariano, Rafał Demkowicz-Dobrzanski and P. Perinotti, in progress

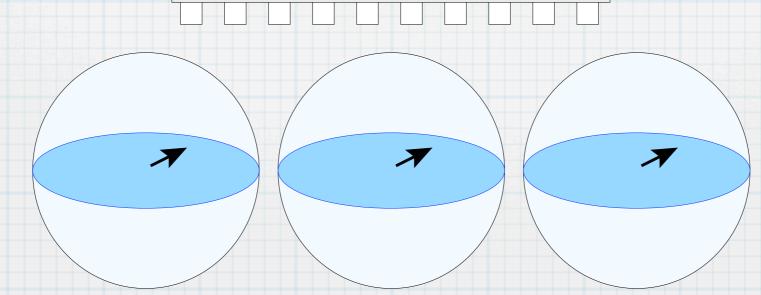
Optimal universal covariant superbroadcasting actually preserves the information about the original input state.



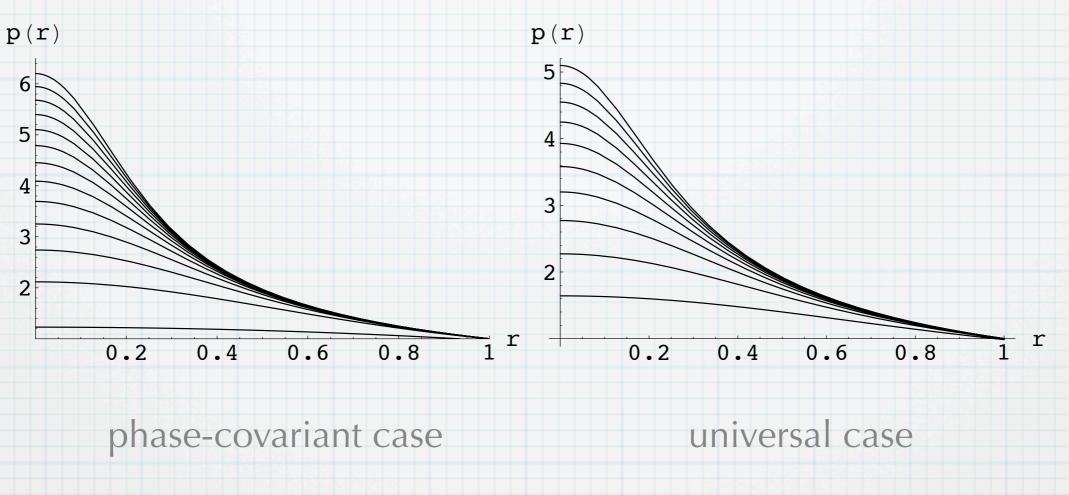
Rafał Demkowicz-Dobrzanski

# Phase-covariant s.b.

#### broadcaster

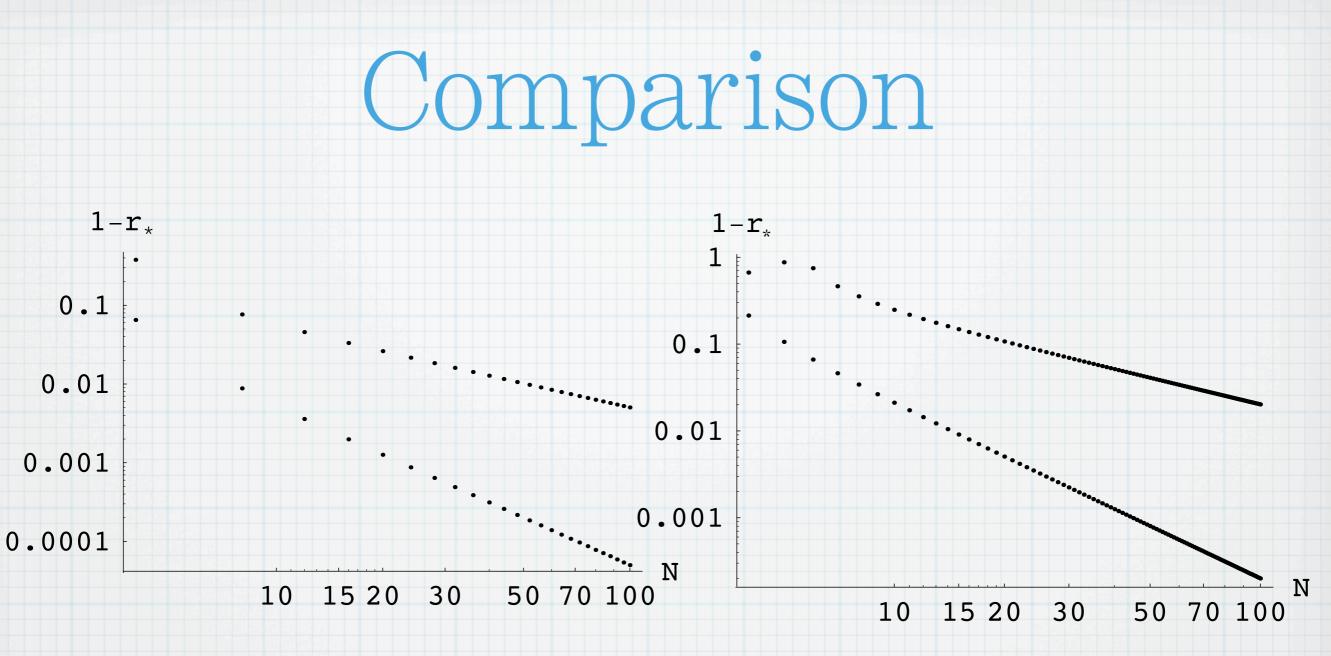


# Comparison



 $M = N + 1, \quad 4 \le N \le 100 \qquad \qquad M = N + 1, \quad 10 \le N \le 100$ 

The purification is higher in the phase-covariant case than in the universal case, since the set of input states is smaller



universal case

 $1 - r_*(N, N+1) \simeq 2N^{-2}$ 

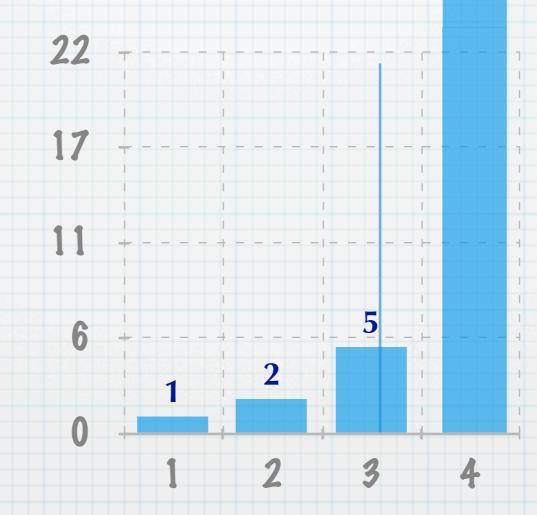
 $1 - r_*(N, \infty) \simeq N^{-1}$ 

phase-covariant case

$$1 - r_*(N, N+1) \simeq \frac{2}{3}N^{-2}$$
$$1 - r_*(N, \infty) \simeq \frac{1}{2}N^{-1}$$

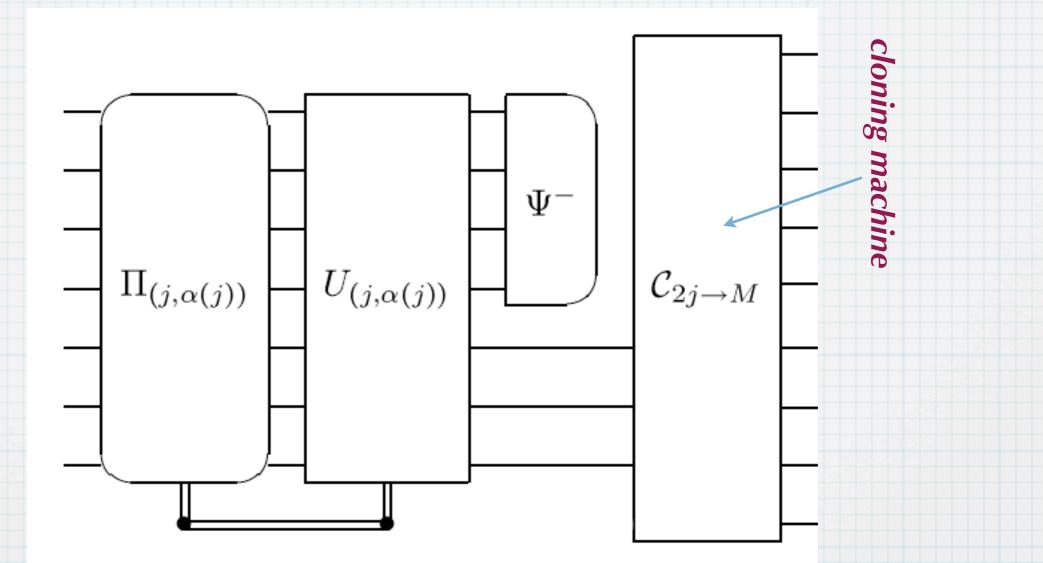
# Phase-covariant s.b.





N

# Realization scheme



 $\Pi_{(j,\alpha(j))} = I_{2j+1} \otimes |\alpha(j)\rangle \langle \alpha(j)| = U_{(l,\chi)}\rho_{(l,\chi)}U^{\dagger}_{(l,\chi)} =$ 

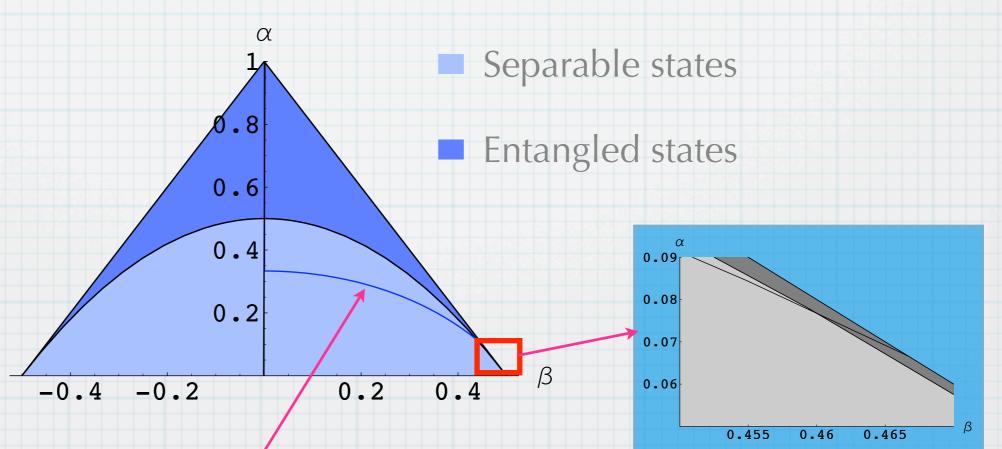
 $\rho_{(l,\chi)} = (r_+ r_-)^{N/2} \left(\frac{r_+}{r_-}\right)^{J_z^{(l)}} \otimes |\chi\rangle\langle\chi|,$ 

 $(r_+r_-)^{N/2} \left(\frac{r_+}{r}\right)^{J_z^{(l)}} \otimes |\Psi^-\rangle \langle \Psi^-|^{\otimes \frac{N-2l}{2}}$ 

Universal broadcasting: symmetric 2-sites output states commuting with  $J_z^{(1)}$ 

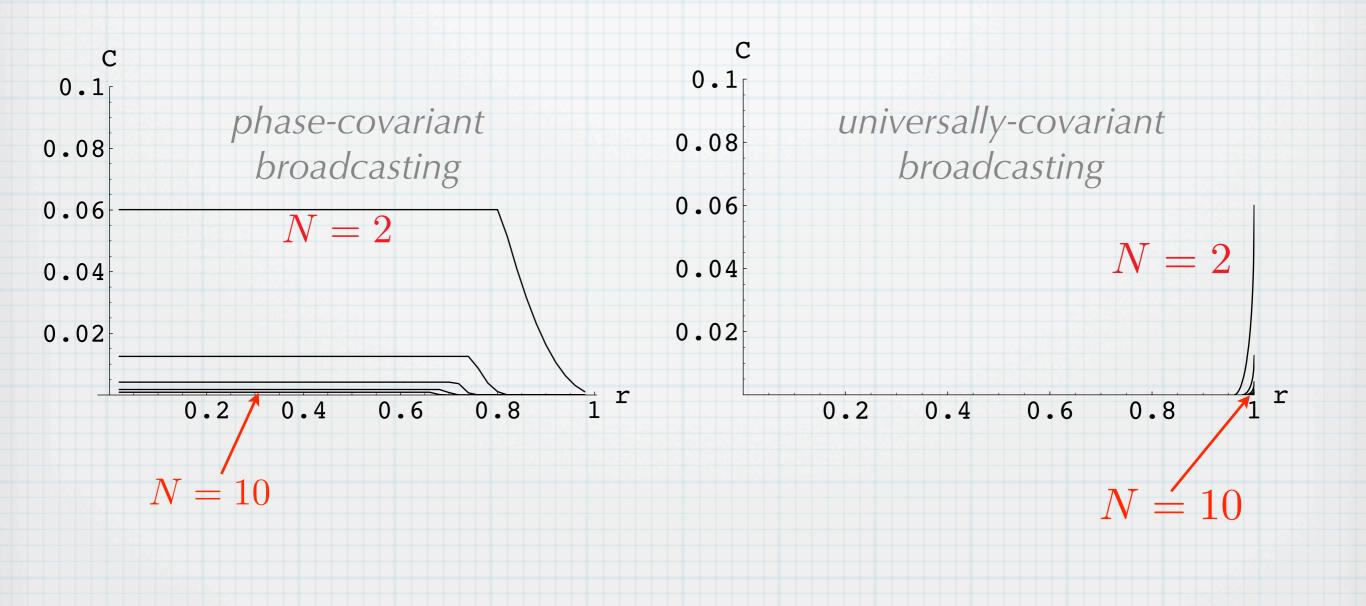
Parametrization of bipartite symmetric states:

$$\rho^{(2)} = \alpha I^{(1)} + \beta J_z^{(1)} + \frac{1 - 3\alpha}{2} J_z^{(1)^2}$$

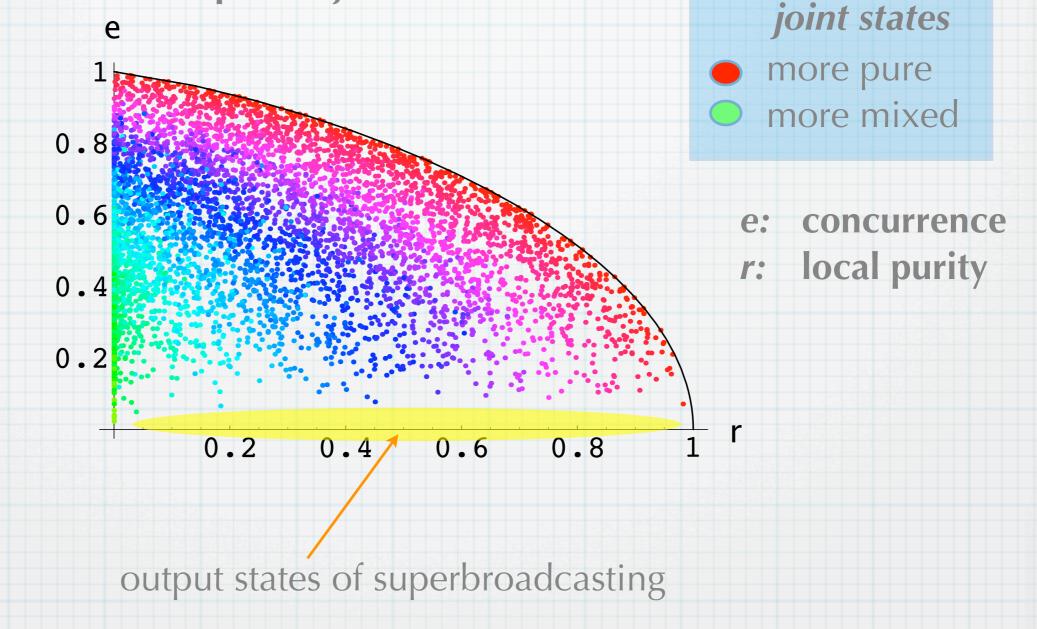


Universal broadcasting output states (4-->5)

Two-sites output *concurrence* C versus the input Bloch vector length r for  $N \rightarrow N + 1$ ,  $2 \le N \le 10$ 



permutation invariant bipartite joint states



#### Is superbroadcasting classical?

- The classical procedure (measurement + preparation) leads only to the same scaling factor as the superbroadcasting for M=∞ (F. Buscemi, G. Chiribella, G. M. D'Ariano, C. Macchiavello, and P. Perinotti, in preparation)
- The protocol for practical achievement of the superbroadcasting map involves Werner cloning map in some stage ----> quantum correlations

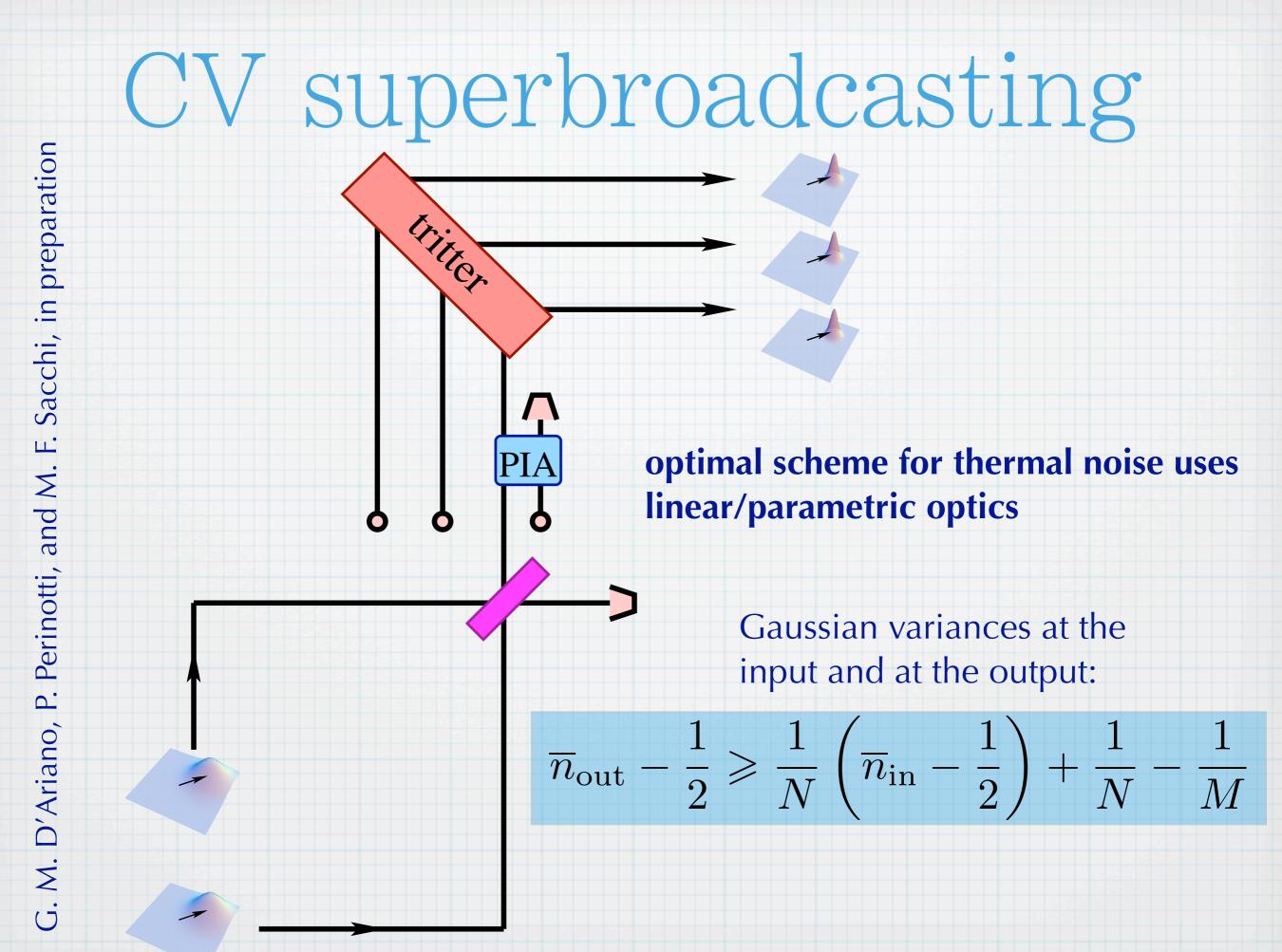
# CV superbroadcasting

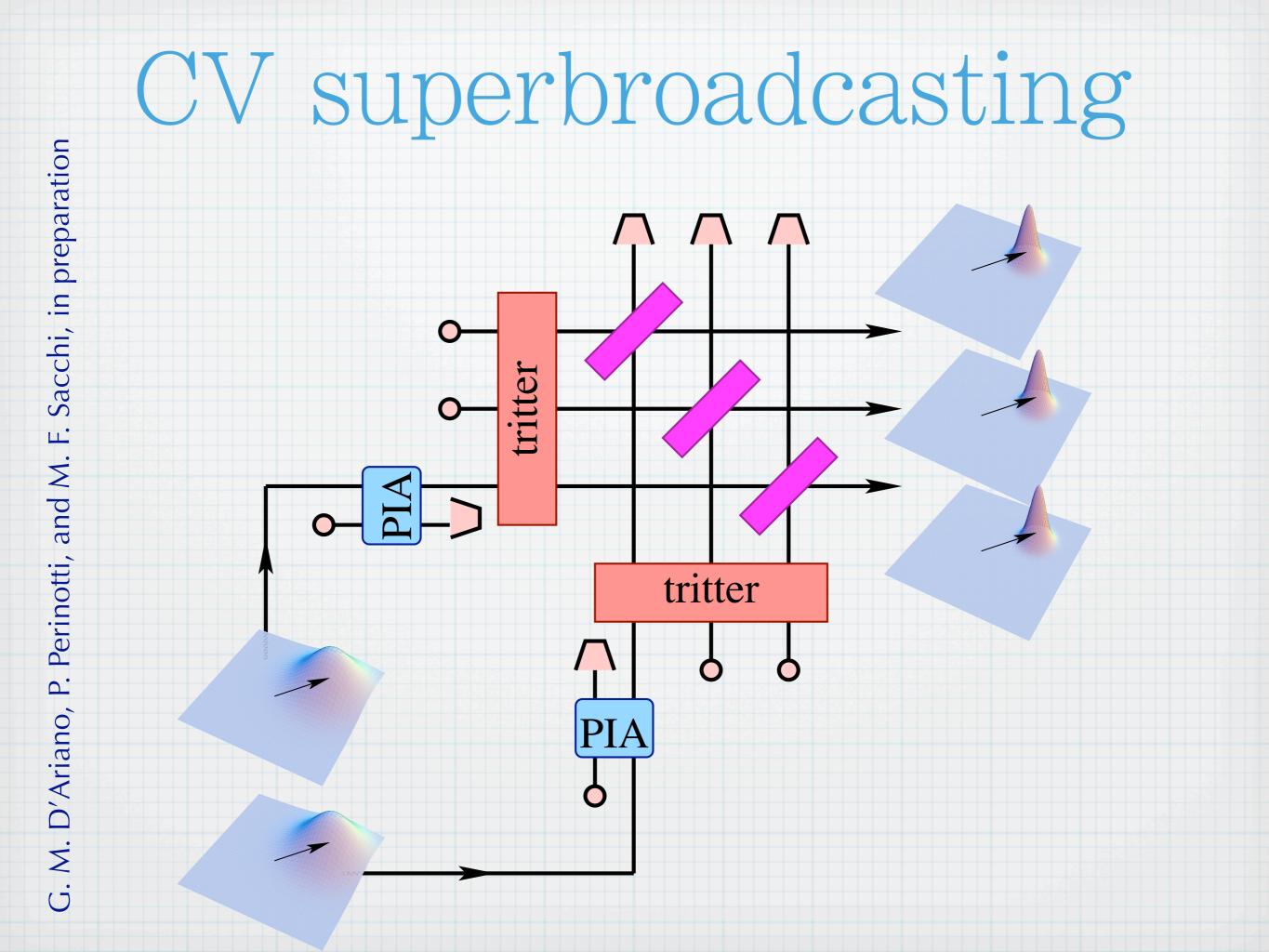
#### superbroadcasting for harmonic oscillators

- feasible for any displaced noisy state
- covariant under the Weyl Heisenberg group of translations on the phase space

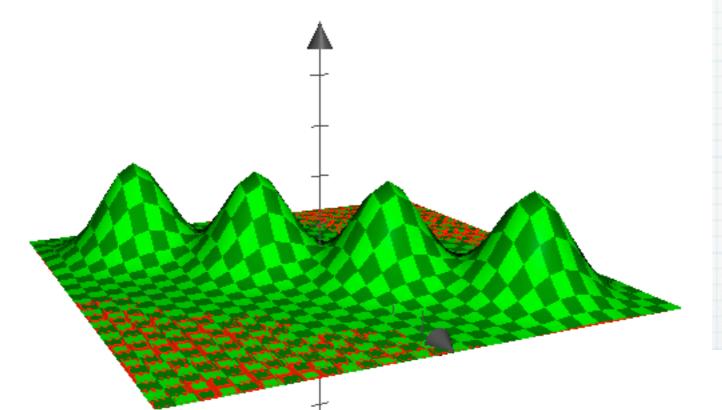


Sacchi



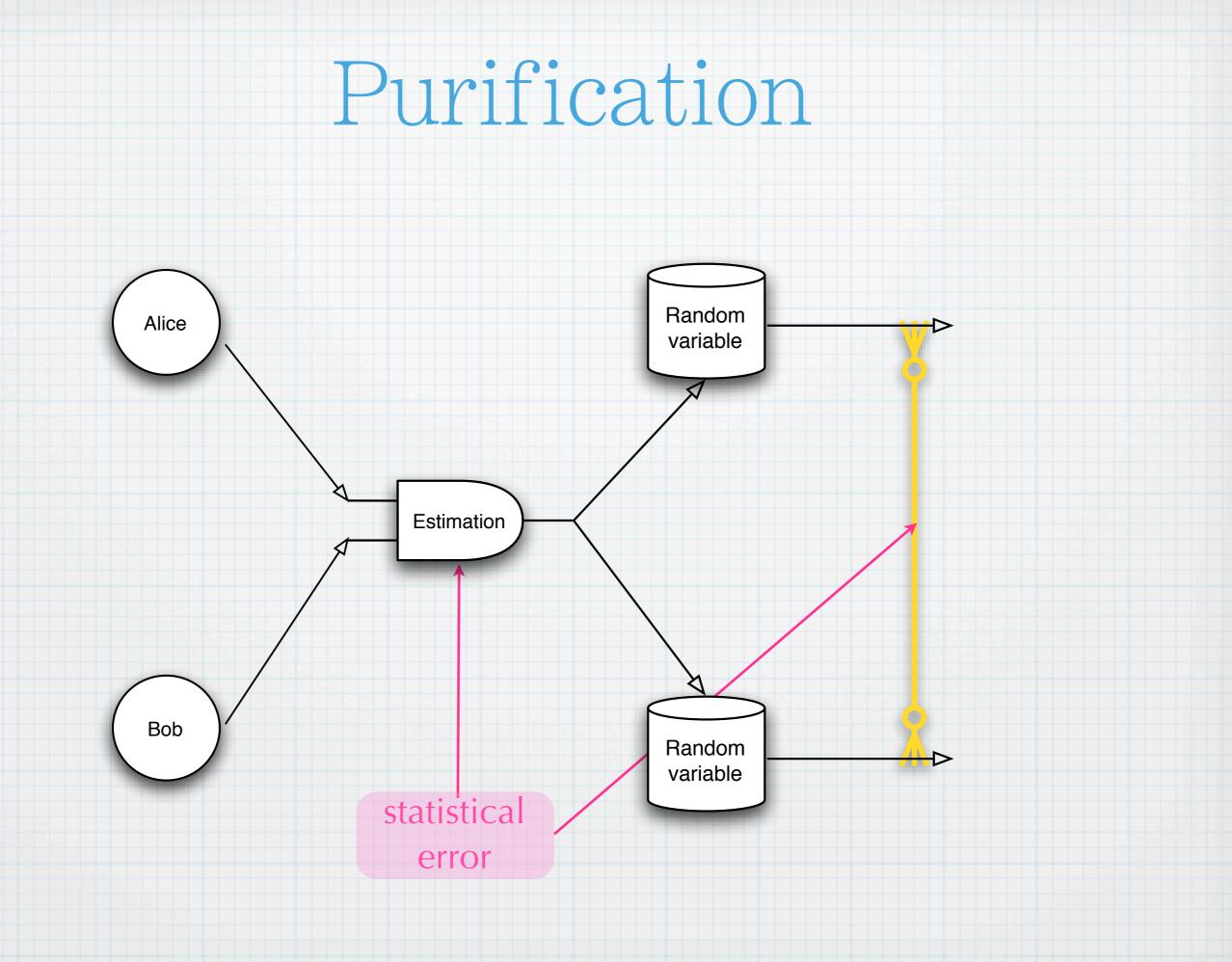


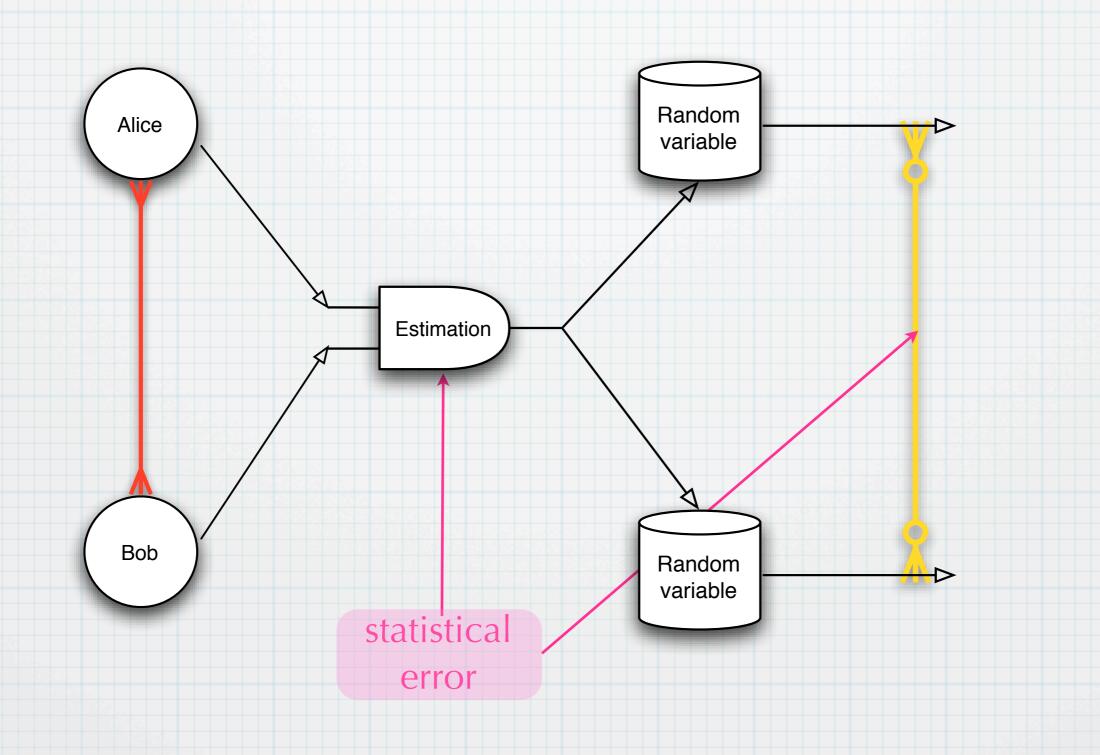
## CV superbroadcasting



Pushing noise into correlations!

reducing thermal noise while creating correlations





• Classical decorrelator for  $p_m(X, Y)$ , X, Y random variables with

$$\langle X \rangle = \langle Y \rangle = m, \quad p_m(X,Y) \neq p_m(X)p_m(Y).$$

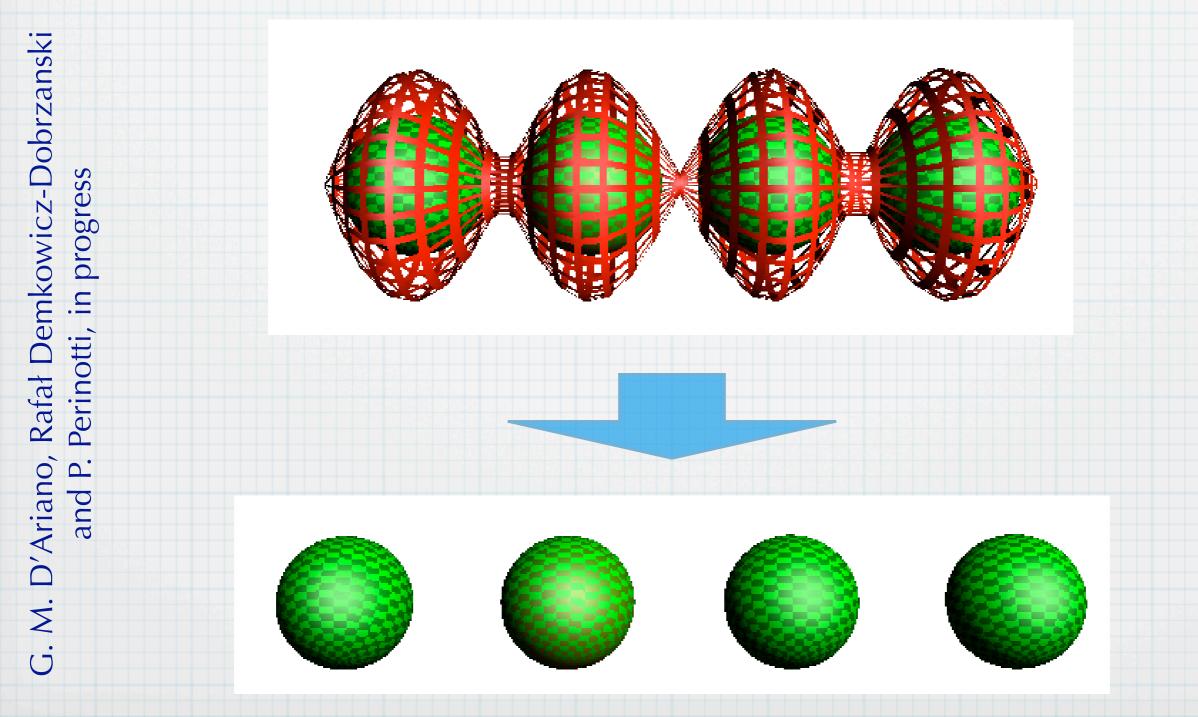
data processing:

$$\forall m \begin{cases} X' = X'(X,Y), & \langle X' \rangle = m \\ Y' = Y'(X,Y), & \langle Y' \rangle = m \end{cases} \qquad p_m(X',Y') = p_m(X')p_m(Y').$$

Quantum decorrelator  $\mathscr{D}$  for  $R \neq \operatorname{Tr}_2[R] \otimes \operatorname{Tr}_1[R]$ 

 $\mathscr{D}(U_g^{\otimes 2}RU_g^{\otimes 2\dagger}) = U_g\rho U_g^{\dagger} \otimes U_g\rho U_g^{\dagger} \ \forall g \in \mathbf{G}.$ 

It is possible *to decorrelate a state* by reducing the purity at each use and/or reducing the number of uses.

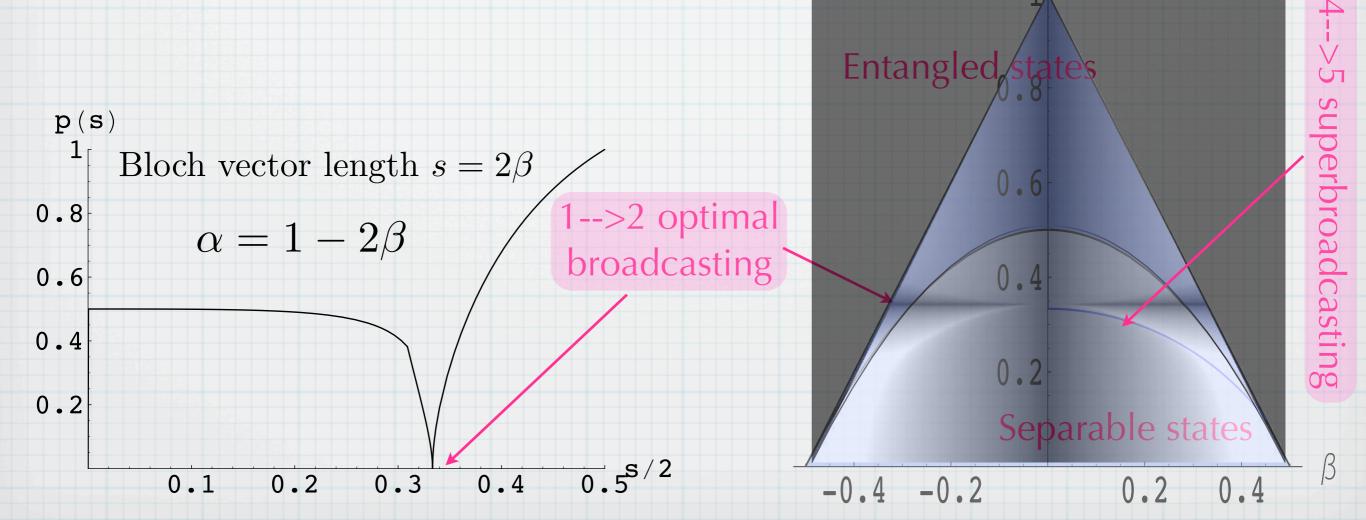


bipartite symmetric states

 Quantum mechanically perfect decorrelation is possible!

$$\rho^{(2)} = \alpha I^{(1)} + \beta J_z^{(1)} + \frac{1 - 3\alpha}{2} J_z^{(1)^2}$$

α

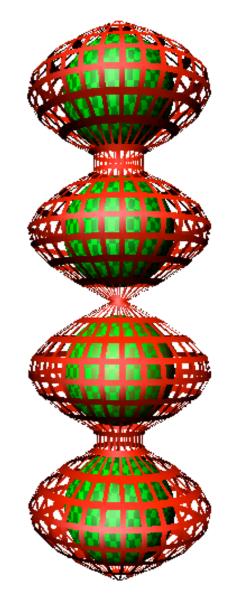


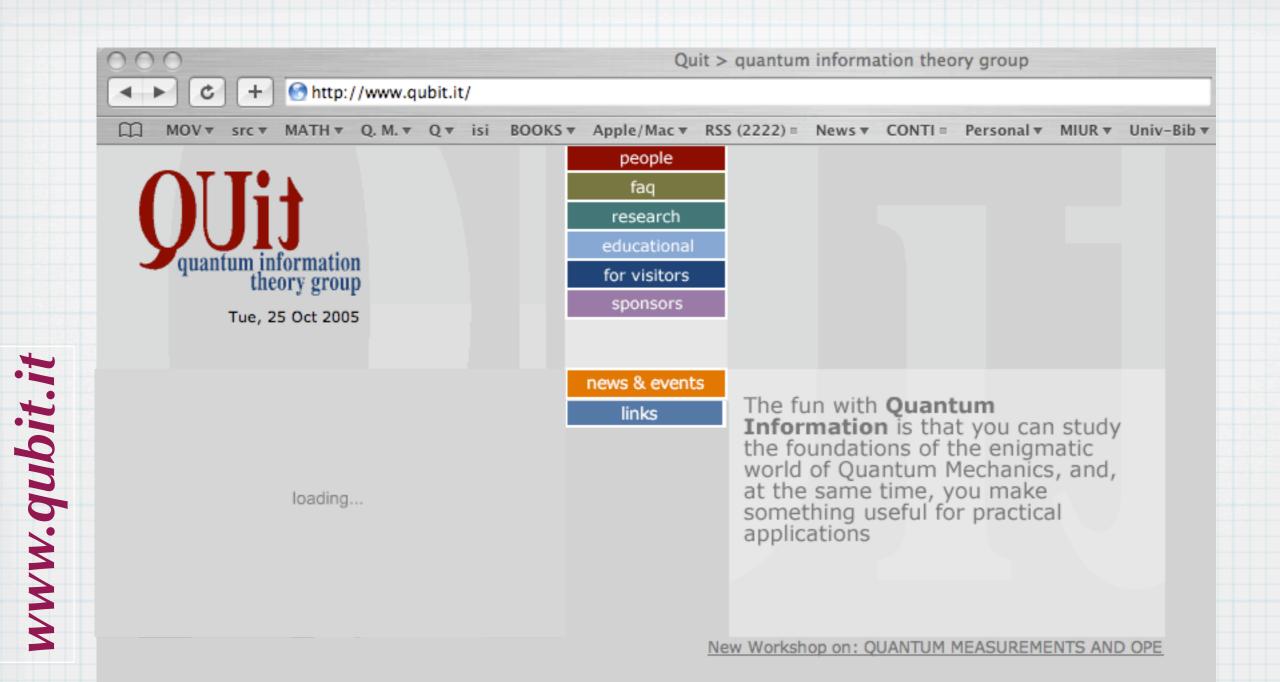
# Marginal estimation

- Quantum mechanically the optimal joint state for estimation of local marginal states is a correlated state [R. Demkowicz-Dobrzanski, Phys. Rev. A 71 062321 (2005)]
- Such joint states can be decorrelated perfectly
- [classically, the optimal joint probability for estimation of marginals is uncorrelated ... ]

## Summary

- It is possible to purify while broadcasting for sufficiently many input copies
- It is easier to superbroadcast starting from larger numbers of input copies and from more mixed states
- The minimum number of input copies depends on the set of input states
- Optimal broadcasting is achieved by a projection followed by a conditioned unitary and a optimal cloning
  - Information on the single-site input state is preserved
  - Superbroadcasting corresponds to pushing the noise of single uses into their correlations
  - CV superbroadcasting is feasible
  - Decorrelation quantum mechanically is possible







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credits